# Chapter 6

# Index of a vector field in the plane

References: Strogatz; Coddington-Levinson, page 398.

Let $\gamma$ be a simple closed curve in the plane, and let $\mathbf{f} = (f_1, f_2)$ be a continuous vectorfield in the plane. Let us traverse the curve in the counterclockwise direction, and let us keep track of what the vector $\mathbf{f}(\mathbf{x})$ in our moving frame. Clearly by the time we (i.e. the point $\mathbf{x}$) come back to our starting point, the vector $\mathbf{f}(\mathbf{x})$ returns to its initial value. The integer number of revolutions the vector $\mathbf{f}(\mathbf{x})$ made is called the index. More precisely

**Definition 6.1.** *Index of a vectorfield* $\mathbf{f}$ *on the curve* $\gamma$ *is the number of turns the vector* $\mathbf{f}(\mathbf{x})$ *makes as the point* $\mathbf{x}$ *traverses* $\gamma$ *exactly once in a counterclockwise direction. More precisely, we parametrize* $\gamma$ *by* $\mathbf{x}(t) = (x(t), y(t))$, $0 \le t \le 1$, *going counterclockwise. Let* $\theta(t) = arg\, \mathbf{f}$ *be the angle formed with the* $x$*–axis, and define*

$$i_\gamma(\mathbf{f}) = \frac{1}{2\pi}(\theta(1) - \theta(0)). \tag{6.1}$$

*This can be rewritten as*

$$i_\gamma(\mathbf{f}) = \frac{1}{2\pi}\oint \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2} = \frac{1}{2\pi}\int_0^1 \frac{\mathbf{f} \wedge \dot{\mathbf{f}}}{\mathbf{f}^2}\, dt, \tag{6.2}$$

*where* $\mathbf{f} = \mathbf{f}(\mathbf{x}(t))$, *provided the curve and the vector field are differentiable.*

**Remarks.** *1. The definition only makes sense if* $\mathbf{f} \neq \mathbf{0}$ *on* $\gamma$*, since the argument of a zero vector is undefined. 2. We insist on continuity of* $\theta(t)$*; without this continuity assumption* $\theta(t)$ *is multiple valued; with this continuity assumption, (6.1) is defined uniquely, independent of the particular choice of* $\theta(t)$*.*

Figure 6.1 gives examples of various indices.

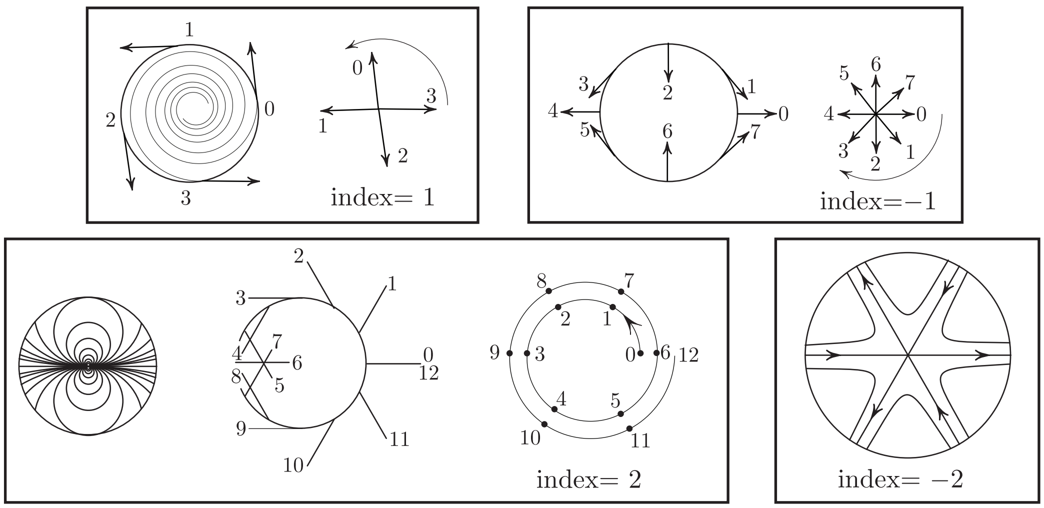


Figure 6.1: Index around a curve surrounding the spiral, the saddle, the dipole and the double saddle.

**Theorem 6.1.** *If there are no equilibria inside and on a closed non-selfintersecting curve* $\gamma$*, then* $i_\gamma(\mathbf{f}) = 0$*.*

The index therefore detects equilibria: a nonzero index implies the presence of equilibria inside $\gamma$. Is the converse of this theorem true?

**Proof of Theorem 6.1.** Referring to Figure 6.2, let us divide the domain $D$ bounded by $\gamma$ into small subdomains, denoting the boundaries of these by $\gamma_n$, $n = 1, 2, \ldots, N$. The domains are chosen to have diameters so small that the argument of $\mathbf{f}$ varies by less than $2\pi$ over each $\gamma_k$.* Thus

$$i_{\gamma_n}(\mathbf{f}) = 0 \quad \text{for all} \quad n = 1, \ldots, N \tag{6.3}$$

since the change of angle over $\gamma_n$ is an integer multiple of $2\pi$ on the one hand, but is strictly less in absolute value than $2\pi$ on the other (by our construction of $\gamma_n$), and hence is zero.

*note that we use $\mathbf{f} \neq \mathbf{0}$ and the continuity of $\mathbf{f}$ here.

But

$$i_\gamma(\mathbf{f}) = \sum_{n=1}^{N} i_{\gamma_n}(\mathbf{f}),$$

as Figure 6.2 explains: when adding up the indices over each $\gamma_n$, the angle changes over shared edges cancel out, leaving only the angle changes over the unshared arcs of $\gamma_n$.

$\diamondsuit$

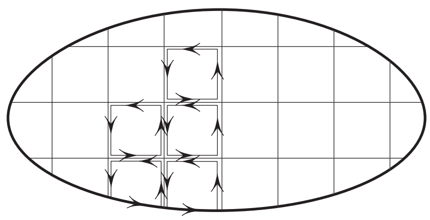

Figure 6.2: Proving that the index is zero if no equilibria inside $\gamma$ are present.

**Theorem 6.2.** *Let* $\mathbf{f}$ *be a continuous vector field, and let* $\gamma_0$, $\gamma_1$ *be two simple closed curves such that one can be continuously deformed into the other without passing through the equilibria of* $\mathbf{f}$*. Then*

$$i_{\gamma_0}(\mathbf{f}) = i_{\gamma_1}(\mathbf{f}) \tag{6.4}$$

**Proof.** Without the loss of generality, let us always parametrize the curves $\gamma_0$, $\gamma_1$ by $t \in [0, 1]$. Let $\gamma(t, \tau)$, $\tau \in [0, 1]$ be a deformation mentioned in the statement of the theorem, i.e. a continuous function from $[0, 1] \times [0, 1] \to \mathbb{R}^2$ such that $\gamma(t, 0) = \gamma_0(t)$ and $\gamma(t, 1) = \gamma_1(t)$.

Since $\mathbf{f}(\gamma(t, \tau)) \neq \mathbf{0}$ for all $0 \leq t, \tau \leq 1$, the argument of $\mathbf{f}$ is also a continuous function of $t, \tau$; we conclude that the index

$$i_{\gamma(\cdot,\tau)} = \frac{1}{2\pi}(\theta(1, \tau) - \theta(0, \tau))$$

is also continuous in $\tau$. But being also integer, the only way for it to be continuous is to be a constant: $i_{\gamma(\cdot,0} = i_{\gamma(\cdot,1}$, i.e. $i_{\gamma_0}(\mathbf{f}) = i_{\gamma_1}(\mathbf{f})$. $\diamondsuit$

The last theorem suggests that what really determines $\gamma$ in $i_\gamma(\mathbf{f})$ is not the particular choice of $\gamma$, but rather the equilibria inside $\gamma$. This leads to the definition of the index of an equilibrium, as follows.

**Definition 6.2.** *The index of an isolated equilibrium point* $P$ *of a vector field is the index of the vector field over a simple closed curve* $\gamma$ *containing* $P$ *and no other equilibrium points in its interior.*

For this definition to make sense we must show that the choice of $\gamma$ in the definition does not matter, i.e. that for any two curves $\gamma_0$, $\gamma_1$ surrounding $P$ and enclosing no other equilibria,

$$i_{\gamma_0}(\mathbf{f}) = i_{\gamma_1}(\mathbf{f}).$$

One way to show this is to argue that one curve can be deformed into the other without passing through equilibria. Rather than proving the existence of such a deformation, let us use a different way: surround the equilibrium inside $\gamma_0$ by a circle that lies entirely inside $\gamma_0$, Figure 6.3, and form a compound closed path as shown in the figure. The shaded region $\Gamma$ (bounded by $\gamma$, the circle and the two segments) encloses no equilibria and thus

$$i_\Gamma(\mathbf{f}) = 0, \tag{6.5}$$

according to Threorem 6.1.* The angle changes over the back–and–forth trip along the cut cancel, and (6.5) becomes

$$i_\gamma(\mathbf{f}) - i_{\gamma_C}(\mathbf{f}) = 0 \tag{6.6}$$

Returning now to two curves $\gamma_0,\ \gamma_1$ surrounding the same equilibrium and no others, we pick the circle lying inside both curves; by (6.6) $i_{\gamma_k}(\mathbf{f}) = i_{\gamma_C}(\mathbf{f})$, $k = 0,\ 1.$ e

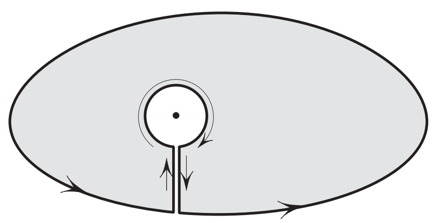

Figure 6.3: The index over $\gamma$ equal the index over a circle.

**Theorem 6.3.** *The index of a vectorfield on a simple closed curve equals the sum of indices of the equilibrium points inside that curve.*

**Proof** is illustrated by Figure 6.4 and is left as an exercise.

**Theorem 6.4.** *The index of the vectorfield tangent to a simple closed curve (and not vanishing on this curve) equals 1.*

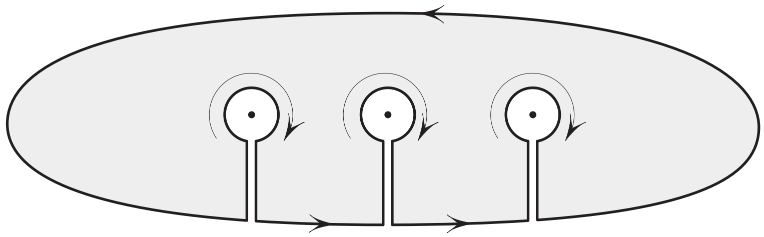

Figure 6.4: Proof of Theorem 6.3.

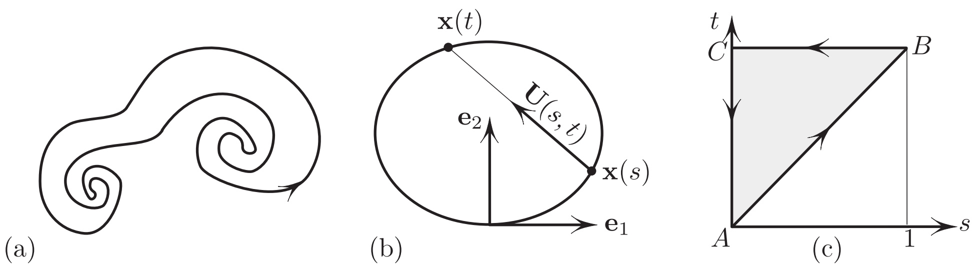


Figure 6.5: Proof of Theorem 6.4.

**Proof.** Although the statement is near–obvious for a convex curve, a glance at a messy curve in Figure 6.5 makes it less clear how to prove the statement.

Let $\mathbf{x} = \mathbf{x}(t), \quad 0 \le t \le 1$ be a parametrization of the curve, which we position so that it lies in the upper half-plane and is tangent to the $x$–axis at the origin, Figure 6.5(b). For an $(s,t)$–chord, consider its *unit direction vector*:

$$\mathbf{U}(s,t) = \frac{\mathbf{x}(t) - \mathbf{x}(s)}{|\mathbf{x}(t) - \mathbf{x}(s)|}, \quad 0 \le s \le t \le 1. \tag{6.7}$$

Actually, $U$ is undefined for $s = t$ and for $(s,t) = (0,1)$ – precisely when $\mathbf{x}(s) = \mathbf{x}(t)$, with $s \le t$. We extend $\mathbf{U}$ by continuity for these values of $s, t$: namely, we define

$$\mathbf{U}(t,t) = \lim_{s \uparrow t} \mathbf{U}(s,t)$$

– this is precisely the unit tangent vector in the positive direction, the vector in whose rotation we are interested. And we let $\mathbf{U}(0,1) = \lim_{t\uparrow 1} \mathbf{U}(0,t) = -\mathbf{e}_1$, the unit vector along the negative $x$–axis. With this definition $\mathbf{U}$ is a vector field defined on the triangle $0 \le s \le t \le 1$ in the $(s,t)$–plane, Figure 6.5(c).

*To make the path in Figure 6.3 non-selfintersecting, as required by Theorem 6.1, we can spread the two lines by a small distance $\varepsilon$, and then take the limit of (6.5) as $\varepsilon \downarrow 0$.

Now the idea is to view $\mathbf{U}(s,t)$ as a vectorfield on the shaded triangle in the Figure (note that we started with a vector field defined only on a curve, and ended up with a vector field defined on a triangle!) Since the unit vectorfield has no zeros,

$$i_{ABCA}(\mathbf{U}) = 0 \tag{6.8}$$

by Theorem 6.1. We conclude that

$$i_{AB}(\mathbf{U}) = i_{AC}(\mathbf{U}) + i_{CB}(\mathbf{U}); \tag{6.9}$$

we use the same notation for the index even though $AB$, $AC$ and $CB$ are not closed curves. But $i_{AC}(\mathbf{U}) = \pi/2\pi = 1/2$, since the vector $\mathbf{U}(0,t)$ starts with $\mathbf{e}_1$, ends with $-\mathbf{e}_1$ and stays in the upper half plane. Similarly, $i_{CB} = 1/2$ since $\mathbf{U}(1,0) = -\mathbf{e}_1$, $\mathbf{U}(1,1) = \mathbf{e}_1$ and $\mathbf{U}(1,s)$ is in the lower half plane for $s \in [0,1]$. This shows that $i_{AB}(\mathbf{U}) = 1$, which is the claim of the theorem, since $\mathbf{U}(t,t)$ is the tangent vector to the curve. ♢

### The Bohl–Brower fixed point theorem

**Theorem 6.5.** *Any continuous map $\mathbf{x} \mapsto \phi(\mathbf{x})$ from a disk $D$ into itself has a fixed point.*

**Proof.** Without the loss of generality, let $D$ be the disk $x^2 + y^2 \leq 1$. Consider the displacement vector $\mathbf{f}(x) = \phi(\mathbf{x}) - \mathbf{x}$. We thus produced a vector field on $D$, and our goal is to show that there exists an equilibrium point $\mathbf{x}_0 \in D$, i.e. the point for which $\mathbf{f}(\mathbf{x}_0) = 0$. Assuming that $\mathbf{f}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D$ (otherwise we are done), the index $i_C\mathbf{f}$ over the boundary circle $C$ is well defined. But since the displacement vector $\mathbf{f}(\mathbf{x})$ points into the disk, we have $i_C(\mathbf{f}) = 1$. Consequently, the vector field $\mathbf{f}$ vanishes at some point inside $D$. ♢

### The fundamental theorem of algebra

**Theorem 6.6.** *Any polynomial has at least one root in the complex plane. (The existence of $n$ roots is then a simple consequence.)*

**Proof** (an outline). The polynomial $P(z) = z^n + a_1 z^{n-1} + \ldots a_n$ with complex $z$ (and with possibly complex coefficients) can be interpreted as a vector field in the plane, and the goal is to show that this vector field has an equilibrium. The main idea is to observe that the index of this over a very large circle is determined by the leading term $z^n$, since this term exceeds all the others combined and thus dictates the number of turns made. More formally, there exists $R$ so large that

$$R^n > |a_1|R^{n-1} + \ldots + |a_n|. \tag{6.10}$$

The existence of such $R$ follows from the fact that the ratio of the right to left sides in (6.10) approaches 0 (and hence is less than 1 for some $R$). The idea now is to deform $P(z)$ into a simpler polynomial $z^n$, by a continuous deformation given by

$$P_s(z) = z^n + s(a_1 z^{n-1} + \ldots + a_n), \tag{6.11}$$

with $s$ going from 1 to 0. By (6.10) the vectorfield $P_s$ never vanishes on $|z| = R$, and thus the index remains constant in $s$. But $i_{|z|=R}(z^n) = n$, and thus $i_{|z|=R}(P(z)) = n \neq 0$. This implies that there exists at least one root of $P$. In fact, we showed that the sum of indices of all equilibria is $n$. ♢

**"You cannot comb a sphere"**

**Theorem 6.7.** *Any continuous vector field on the sphere (i.e. a function which attaches to each point on the sphere a tangent vector at that point) has at least one critical point.*

**Proof.** Recall the definition of the stereographic projection, Figure 6.6: a point $N$ on the sphere is singled out, and any other $a$ is mapped to the point of intersection of the line $Na$ with the tangent plane at $S$, the antipode of $N$. In other words, $A$ is the shadow of $a$ with the source of light at $N$. If a vector field is defined on the sphere, i.e. if each point has a velocity, the shadows' velocities are thereby defined as well, and so a vector field on the sphere defines a vector field in the plane. And the equilibrium of one vector field corresponds to an equilibrium of the other. This last remark will reduce the problem of proving existence of equilibria on the sphere to the problem in the plane.

Let $N$ be a nonsingular point on the sphere (if such doesn't exist, then we have an everywhere zero vector field and there is nothing left to prove). Treat $N$ as the north pole, and surround it with a parallel. Figure 6.7 shows the top view of the vector field (i.e. the projection of this small spherical cap $S_+$, and of the vector field through its boundary, onto the tangent plane at $N$.)

Figure 6.6 shows the top view of the vector field on the spherical cap, and the vector field in the plane, resulting from stereographic projection.

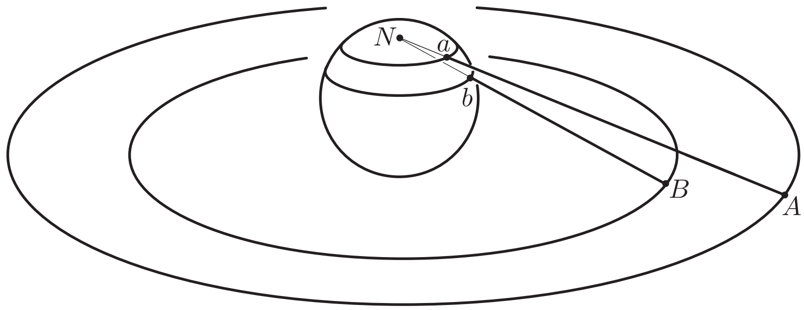


Figure 6.6: Stereorgaphic projection reflects the direction of the field around the tangent to the parallel.

Speaking intuitively at first, the stereographic projection flips the band between two parallels in Figure 6.6 inside out, i.e. the inner ring (as viewed by the observer on the North pole) maps to the outer ring after projection.* Putting it more formally, if a bug, following the vector field, leaves the top cap $S_+$, he enters the south cap $S_-$, and thus his stereographic shadow enters the disk $D$ which is the stereographic projection of $S_-$.

In short, *outward pointing vectors on the boundary of $S_+$ map to inward pointing vectors on the boundary of $D$.* This fact, along with the fact that the index around the parallel is zero, allows one to conclude that the index around the boundary of $D$ is 2, as Figure 6.7) suggests. I omit the details of the proof (the idea is to first count full flips of the vector field in question *relative to the tangent vector to the circle*; the answer turns out to be 1. But the tangent itself makes 1 flip, so the true number of flips is $1 + 1 = 2$.) $\diamondsuit$

**Problem 6.1.** Consider two vector fields $\mathbf{f}$ and $\mathbf{g}$ in $\mathbb{R}^2$. Show that if the angle $\angle(\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})) < \pi$ for all $\mathbf{x}$ on a simple closed curve $\gamma$, then

$$i_\gamma \mathbf{f} = i_\gamma \mathbf{g}$$

**Problem 6.2.** Consider two vector fields $\mathbf{f}$, $\mathbf{g}$ satisfying $|\mathbf{g}(\mathbf{x})| < |\mathbf{f}(\mathbf{x})|$ for all $\mathbf{x}$ on a simple closed curve $\gamma$. Show that

$$i_\gamma \mathbf{f} = i_\gamma (\mathbf{f} + \mathbf{g})$$

*This is only from the point of view of $N$: from the point of view of the South pole, no inversion happens: for the South Pole observer, both $b$ and $B$ lie on the inner rings.

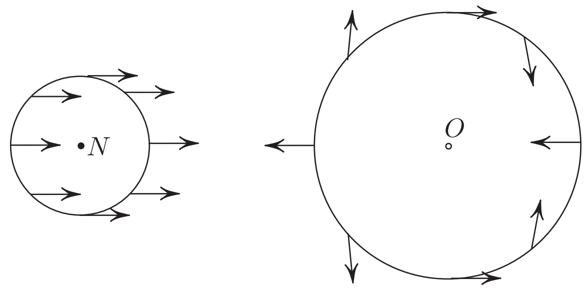


Figure 6.7: Top view of the spherical cap $S_+$ near the north pole (left); disk $D$, the stereographic projection of the complementary south of the parallel cap $S_-$ (right).

**Problem 6.3.** Let $\gamma$ be a closed orbit on an f ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}$ in $\mathbb{R}^2$. Show that a closed orbit of an autonomous ODE in $\mathbb{R}^2$ cannot enclose a saddle equilibrium and no others.