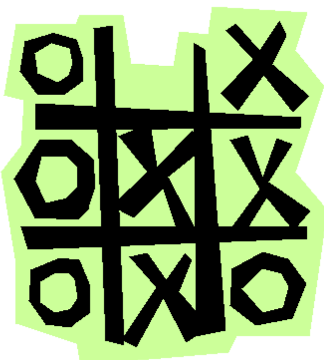
Combinatorial Games

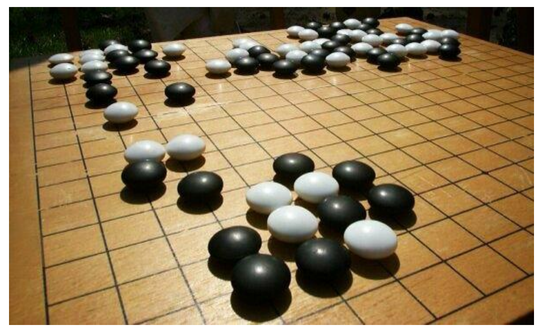
Sequential games

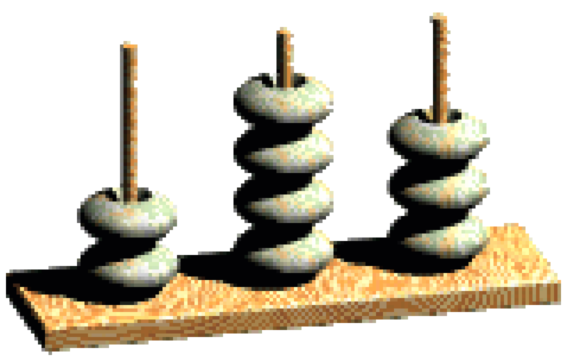
A sequential game is a game where one player chooses his action before the others choose their.

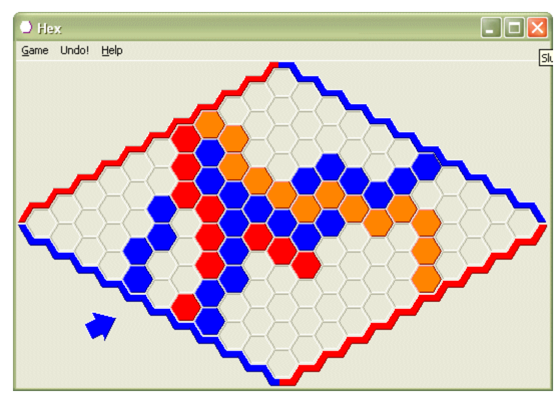
We say that a game has perfect information if all players know all moves that have taken place.

Sequential games









Combinatorial games

- Two-person sequential game
- Perfect information
- The outcome is either of the players wins
- The game ends in a finite number of moves

Combinatorial games

Terminal position: A position from which no moves is possible

Impartial game: The set of moves at all positions are the same for both players

Partizan game: Players may have different possible moves at a given position

Normal play rule: The last player to move wins

Misere play rule: The last player to move loses

Combinatorial games

We will study impartial games with normal play rule only.

Winning strategy

In a two-person combinatorial game, exactly one of the players has a winning strategy.

Zermelo's theorem

In any finite sequential game with perfect information, at least one of the players has a drawing strategy. In particular if the game cannot end with a draw, then exactly one of the players has a winning strategy.

Negation of quantifiers

for logic statements

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

Negation of quantifiers

More generally

$$\neg\forall x_1 \exists y_1 \cdots \forall x_k \exists y_k P(x_1, y_1, \cdots, x_k, y_k)$$
$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \neg P(x_1, y_1, \cdots, x_k, y_k)$$

Proof of Zermelo's theorem

x_i: i^{th} move of 1^{st} player
y_j: j^{th} move of 2^{nd} player

$$\neg 2^{nd}\ \textit{player has winning strategy}$$
$$\Leftrightarrow \neg \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \left(2^{nd}\ \textit{player wins}\right)$$
$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \neg \left(2^{nd}\ \textit{player wins}\right)$$
$$\Leftrightarrow \exists x_1 \forall y_1 \cdots \exists x_k \forall y_k \left(1^{st}\ \textit{player wins}\right)$$
$$\Leftrightarrow 1^{st}\ \textit{player has winning strategy}$$

Subtraction game

- Let n be a positive integer and
  $S \subset \{1,2,3,\cdots n\}$
- There is a pile of n chips.
- A move consists of removing k chips from the pile where $k \in S$.
- The player removes the last chip wins.

Subtraction game

Example when $n = 21$ and

$$S = \{1,2,3\}$$

1. Who has the winning strategy?
2. What is the winning strategy?

Subtraction game

1. Who has the winning strategy?

   Answer:

   When *n* is not a multiple of 4, the first player has a winning strategy. Otherwise the second player has a winning strategy.

Subtraction game

2. What is the winning strategy?

   Answer:

   To remove the chips so that the remaining number of chips is a multiple of 4.

How to find winning strategy?

P-position

The previous player has a winning strategy.

N-position

The next player has a winning strategy.

P-position and N-position

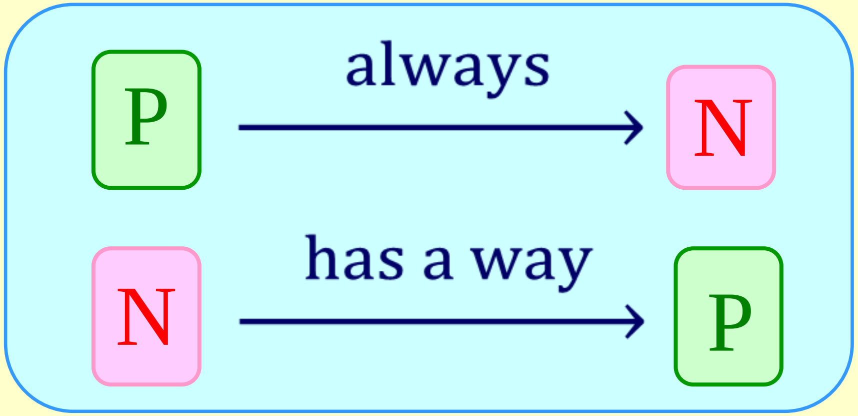
In normal play rule, the player makes the last move wins. In this case,

1. Every terminal position is a P-position
2. A position which can move to a P-position is an N-position
3. A position which can only move to an N-position is a P-position

P-position and N-position

P: previous player has winning strategy

N: next player has winning strategy



Subtraction game

For subtraction game with

$$S = \{1, 2, 3\}$$

Subtraction game

1. Every terminal position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P												

Subtraction game

A position which can move to a P-position is an N-position

0 1 2 3 4 5 6 7 8 9 10 11 …

P N N N

Subtraction game

A position which can only move to an N-position is a P-position

0 1 2 3 4 5 6 7 8 9 10 11 …

P N N N P

Subtraction game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N					

Subtraction game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N	P				

Subtraction game

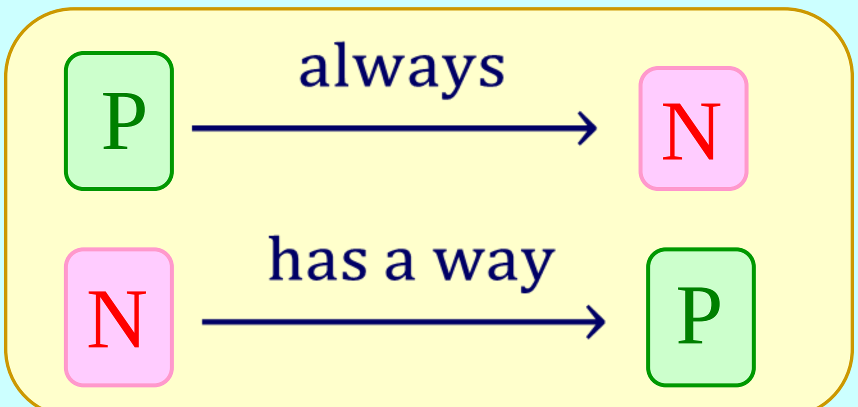
A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	N	N	P	N	N	N	P	N	N	N	…

Subtraction game

P = { 0, 4, 8, 12, 16, 20,… }

N = { not multiple of 4 }



Subtraction game

For subtraction game with

$$S = \{1,3,4\}$$

Subtraction game

1. Every terminal position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P												

Subtraction game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N		N	N								

Subtraction game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N								

Subtraction game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N						

Subtraction game

A position which can only move to an N-position is a P-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P					

Subtraction game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P	N		N	N	

Subtraction game

A position which can move to a P-position is an N-position

0	1	2	3	4	5	6	7	8	9	10	11	…
P	N	P	N	N	N	N	P	N	P	N	N	

Subtraction game

$$P = \{\, 0, 2, 7, 9, 14, 16, \ldots \,\}$$

$$= \{k : k \equiv 0,2 \pmod 7\}$$

$$N = \{\, 1, 3, 4, 5, 6, 8, 10, 11, \ldots\}$$

$$= \{k : k \equiv 1,3,4,5,6 \pmod 7\}$$

Proof of P-positions

To prove that a set P is the set of P-position of a game, we need to do the following.

1. Prove that all terminal positions are in P.
2. Prove that any position in P can only move to a position not in P.
3. Prove that any position not in P has a way to move to a position in P.

Wythoff's game

- There are 2 piles of chips
- On each turn, the player may either
  (a) remove any positive number of chips from one of the piles or
  (b) remove the same positive number of chips from both piles.
- The player who removes the last chip wins.

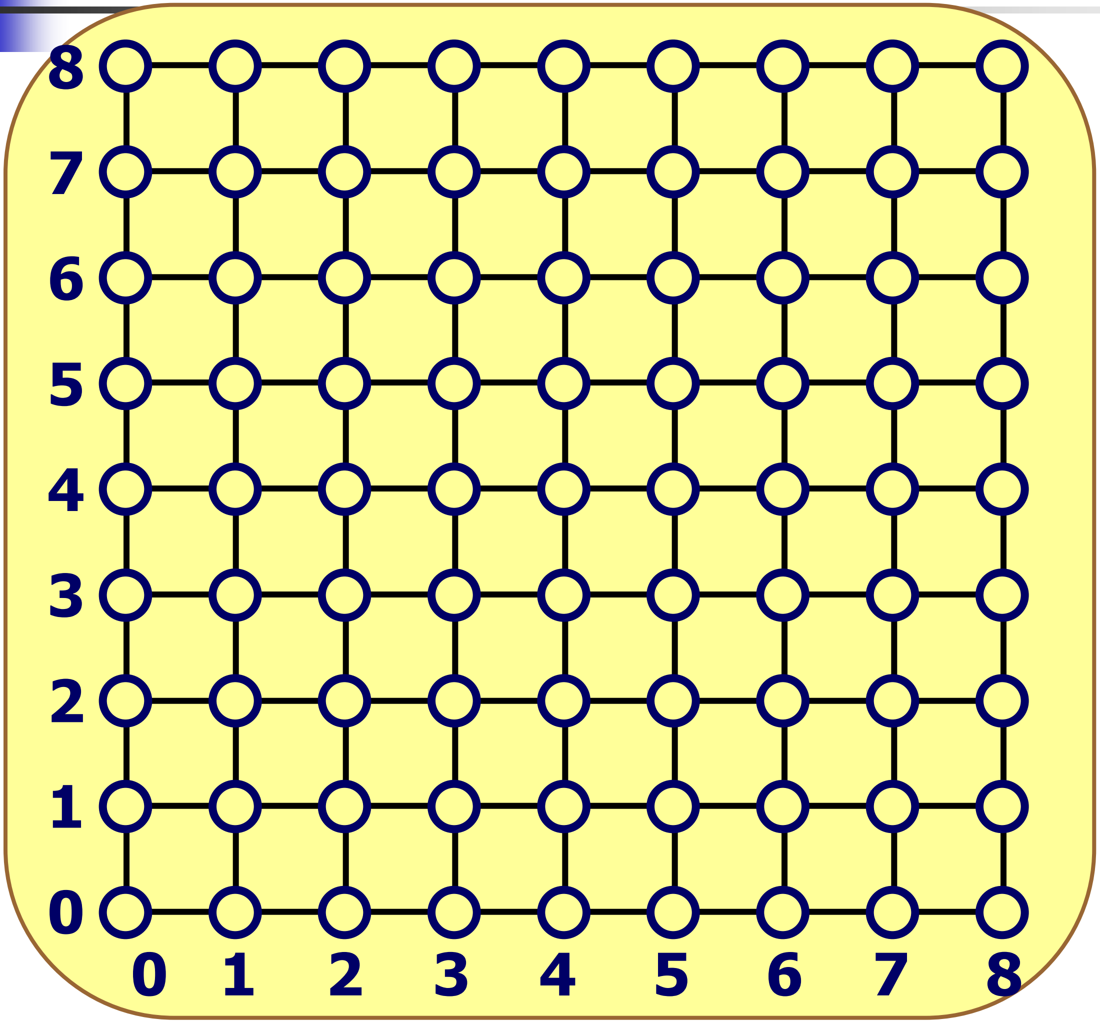
Wythoff's game

P-positions:

{ (0,0), (1,2), (3,5), ?,… }

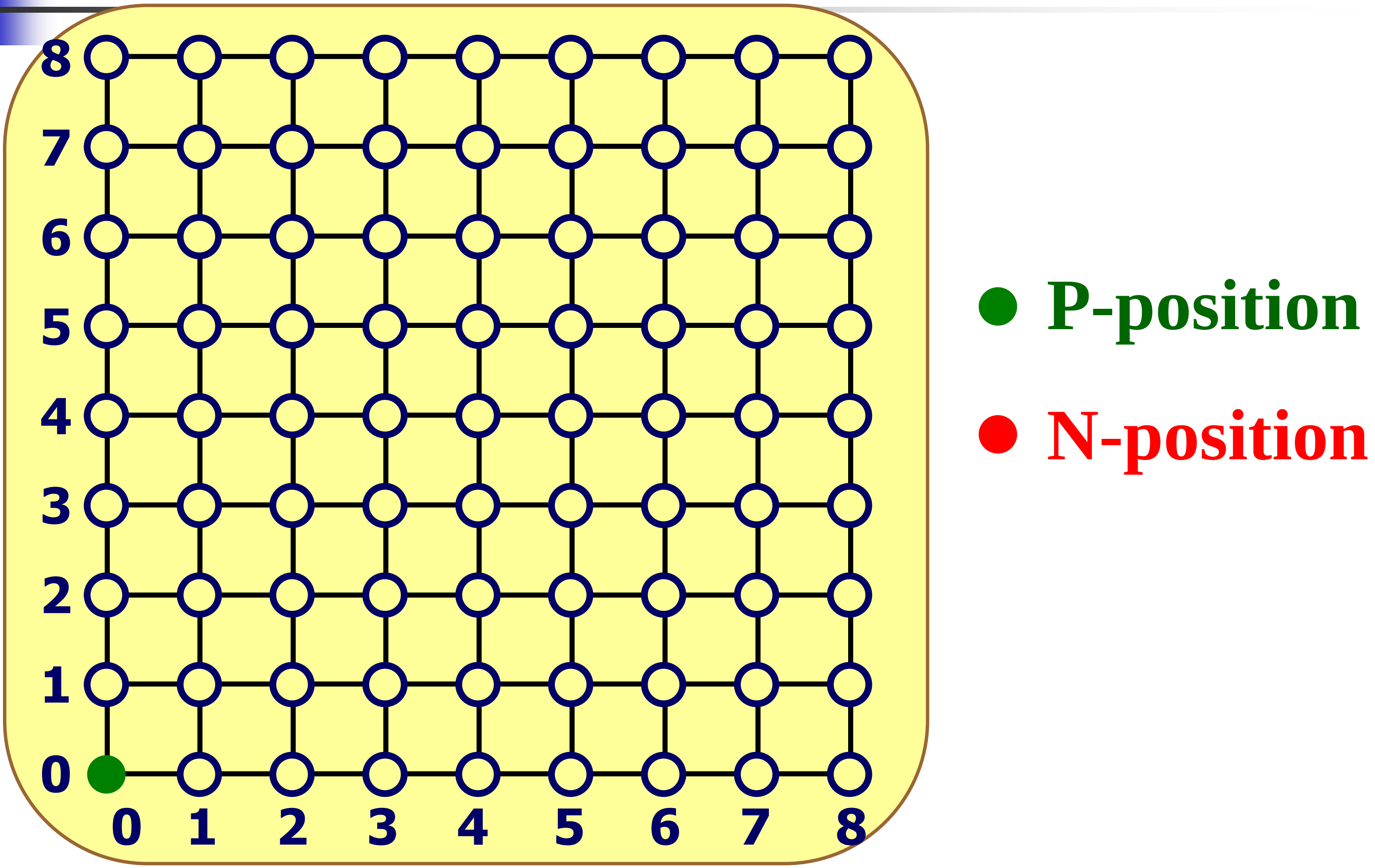
What is the next pair?

Two piles take-away game

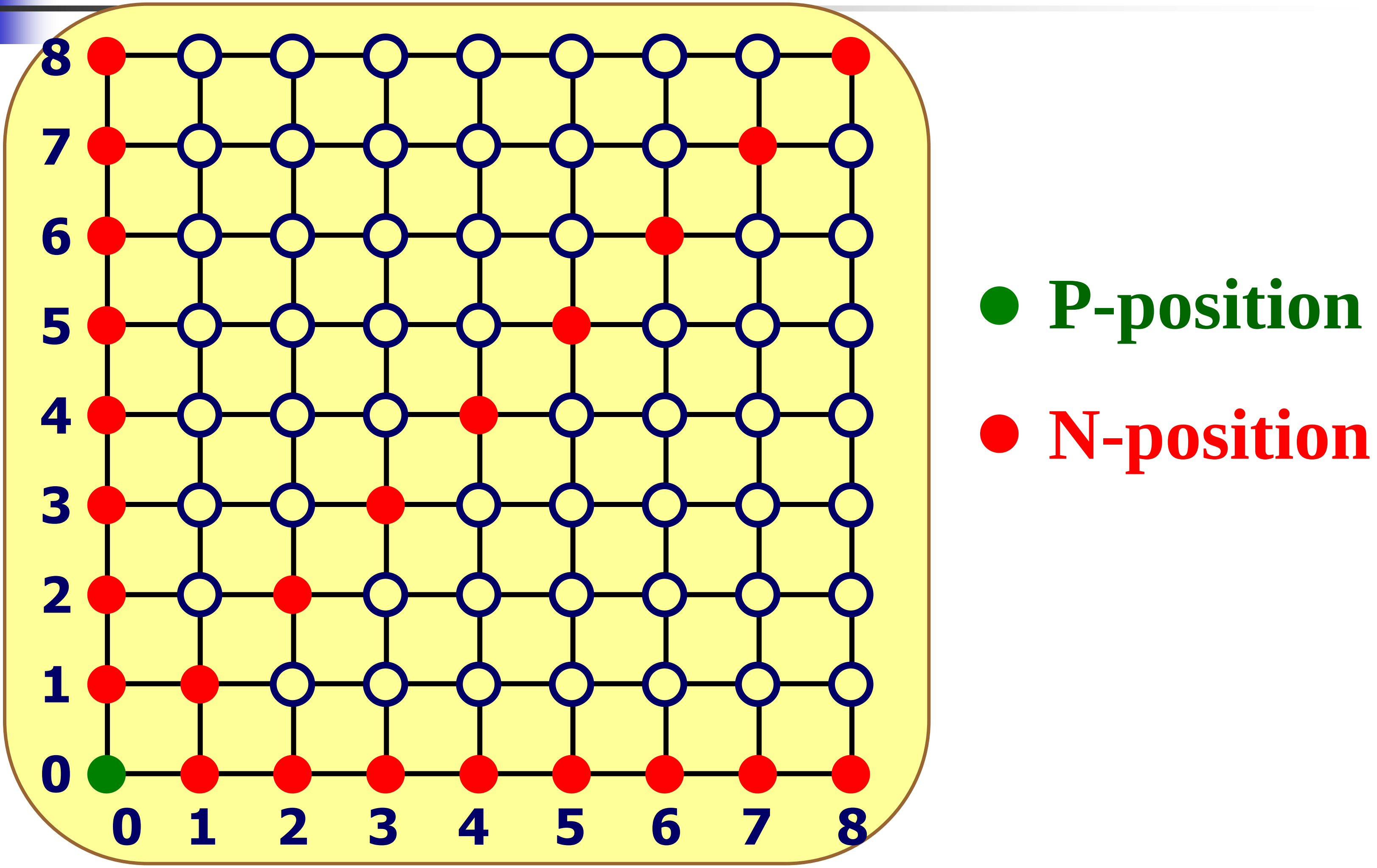


- P-position
- N-position

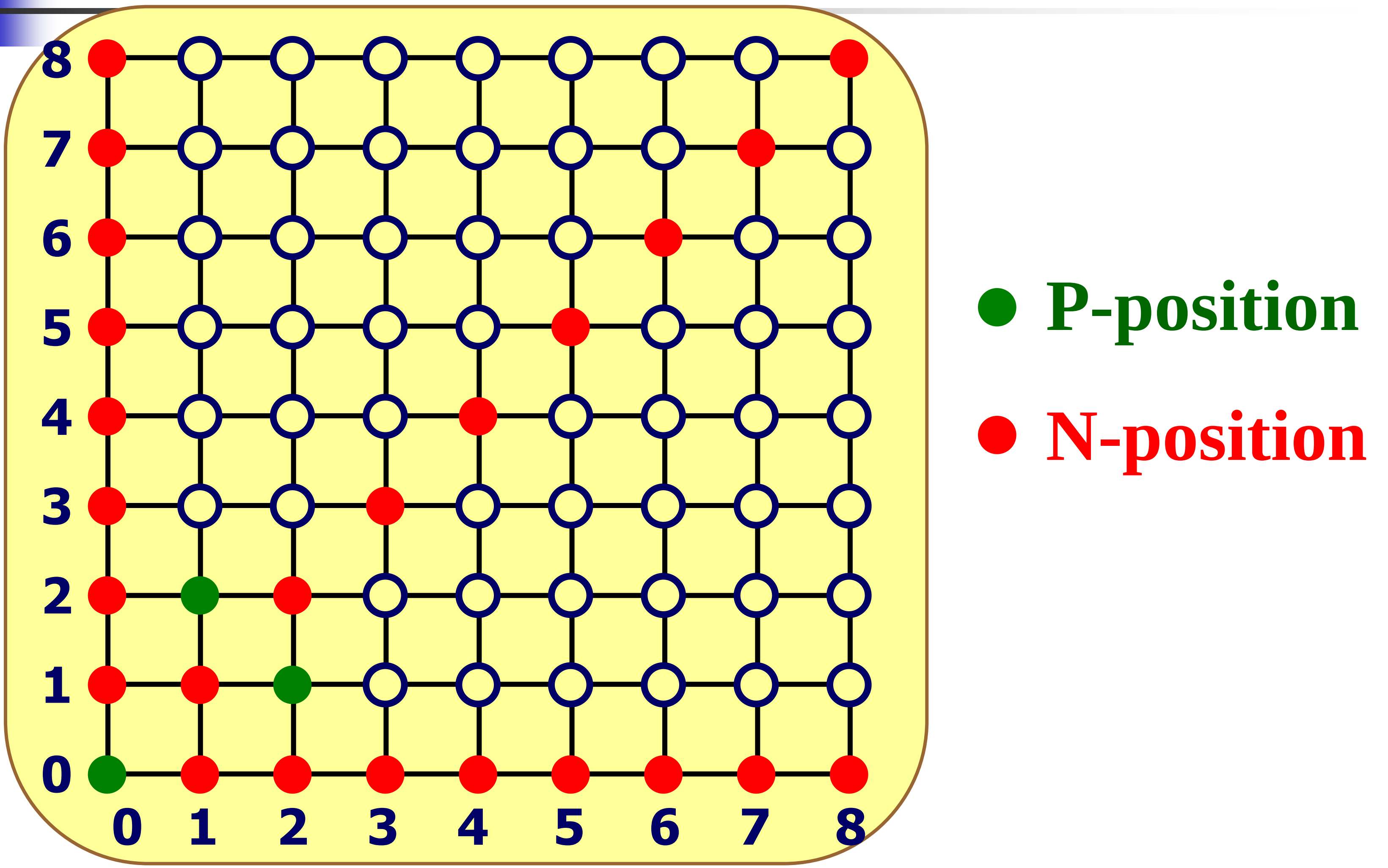
Terminal positions are P-positions



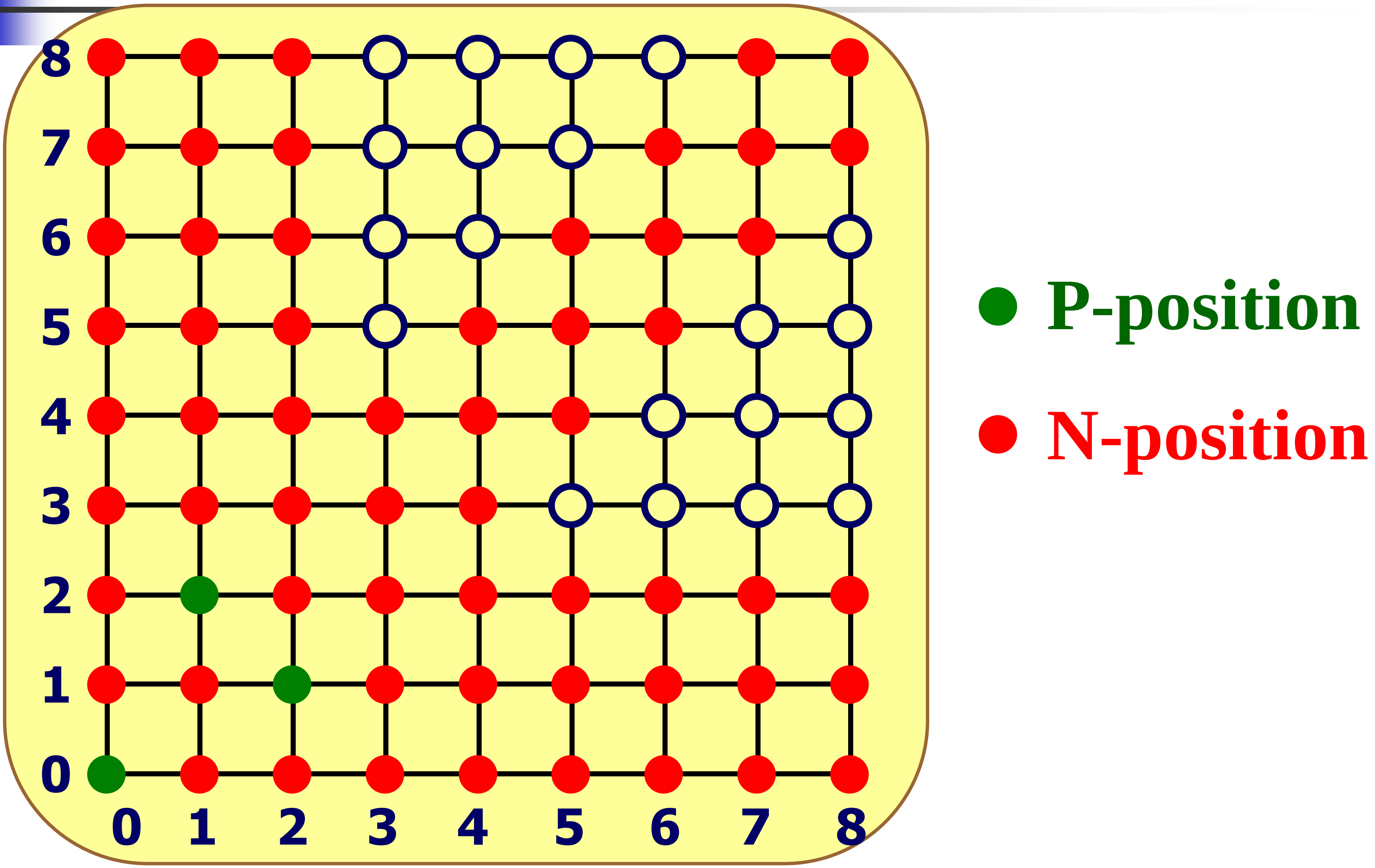
Positions which can move to P-positions are N-positions



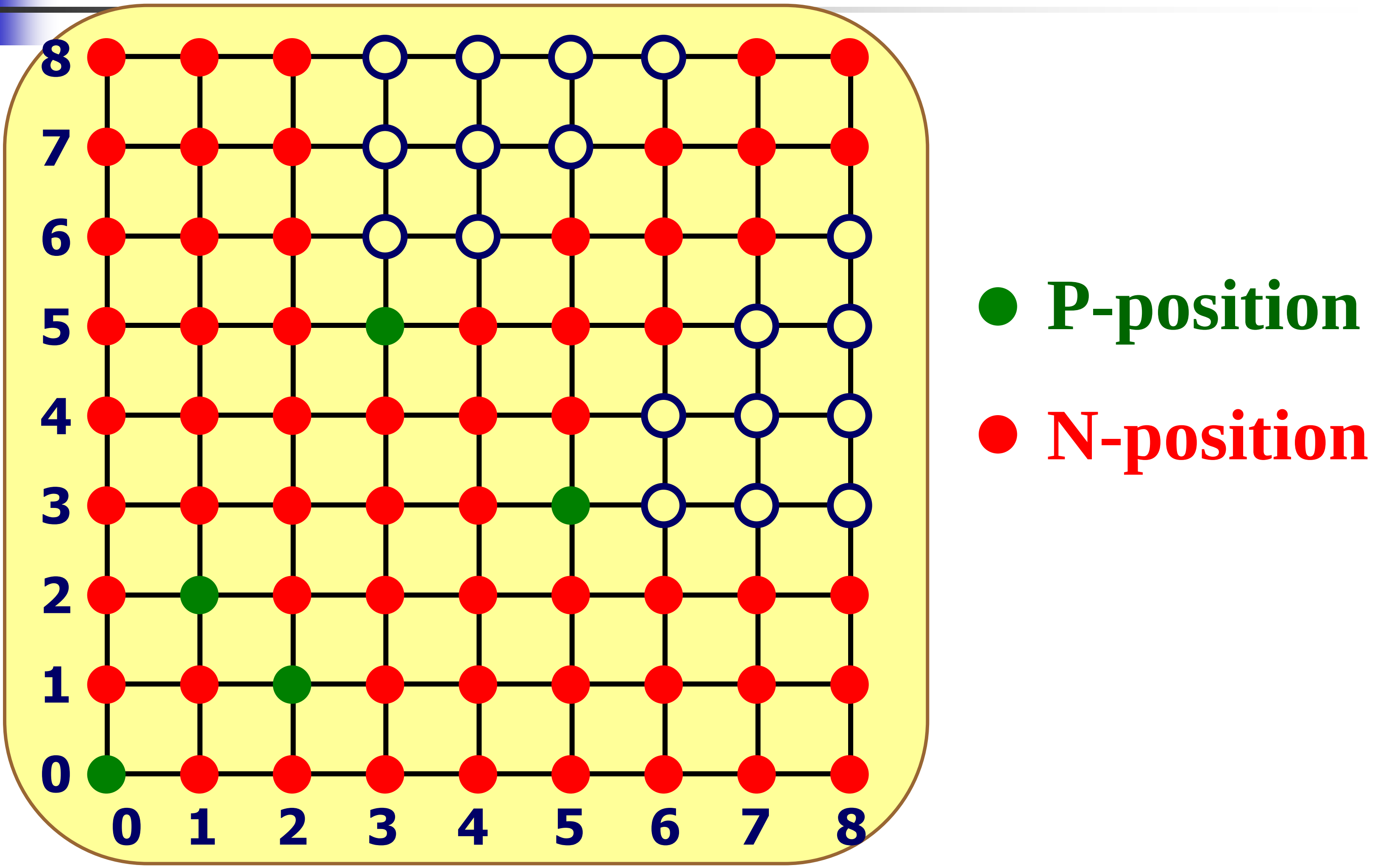
Positions which can only move to N-positions are P-positions



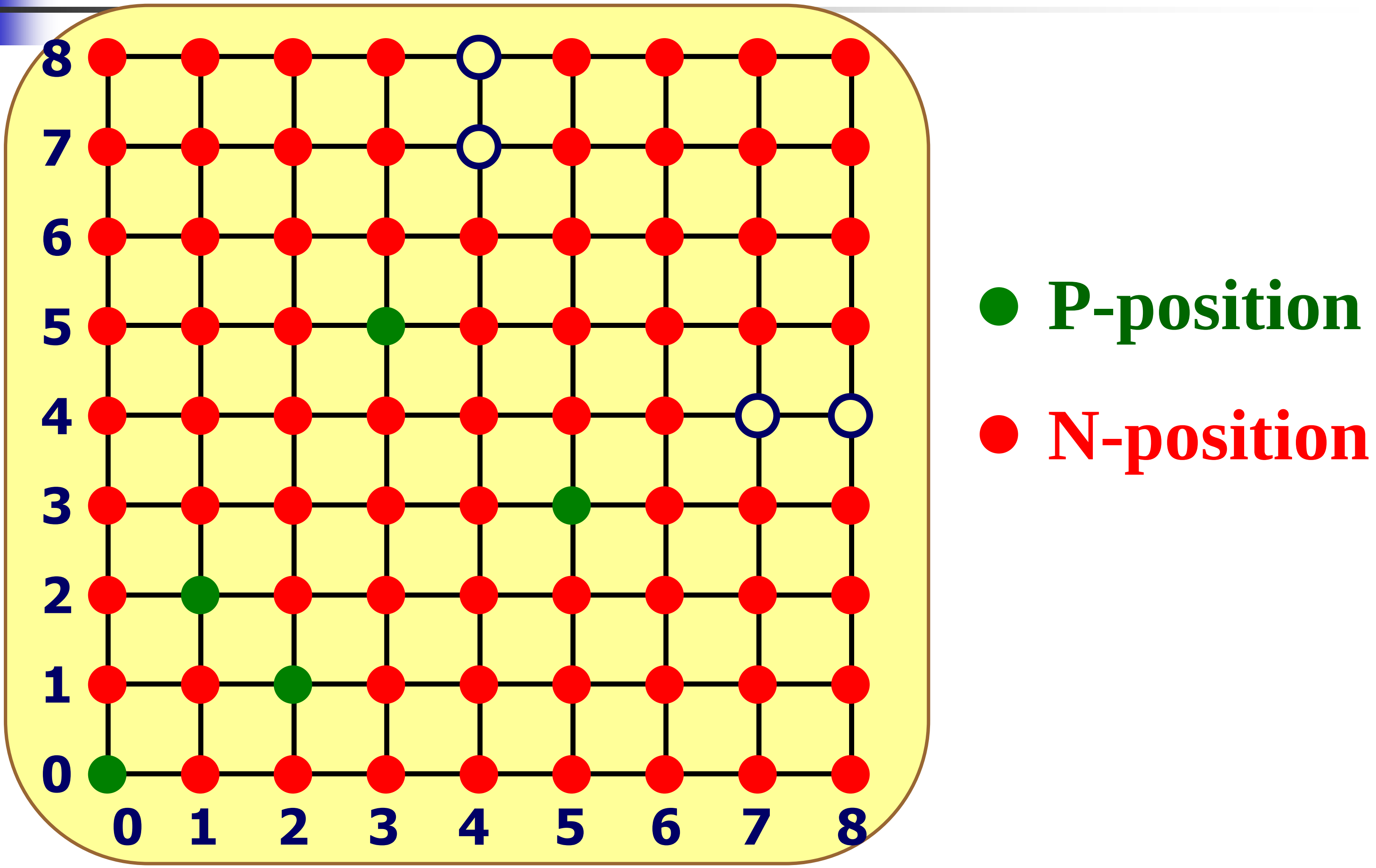
Positions which can move to P-positions are N-positions



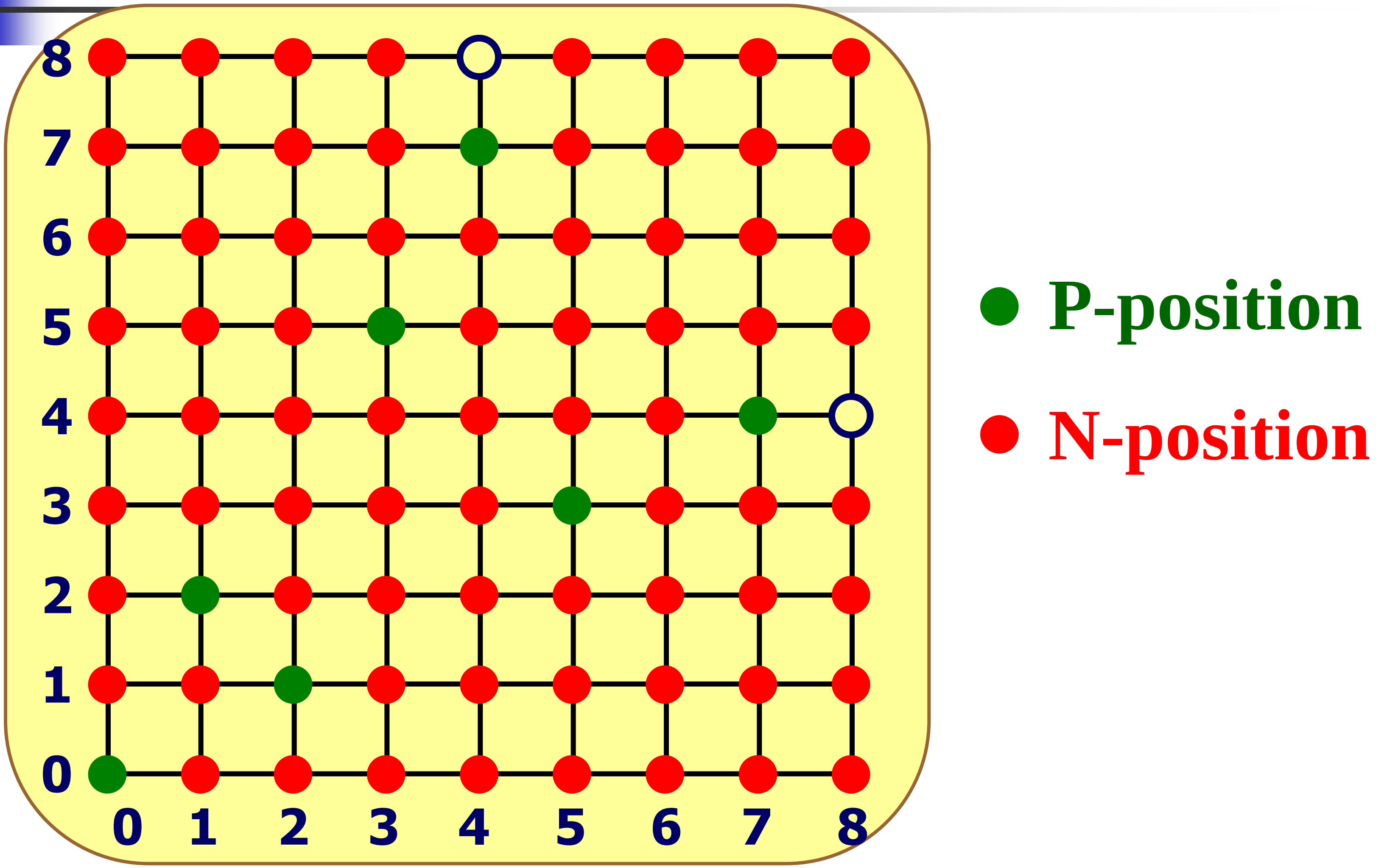
Positions which can only move to N-positions are P-positions



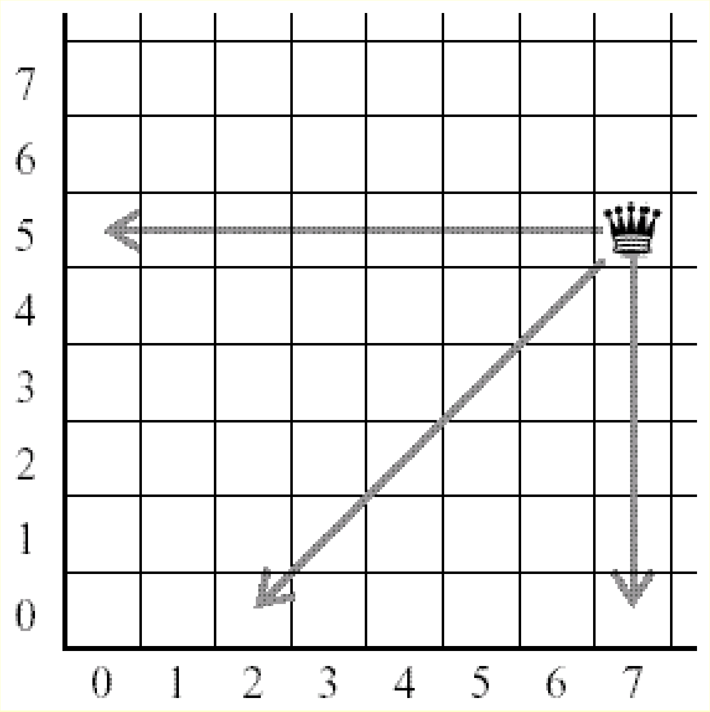
Positions which can move to P-positions are N-positions

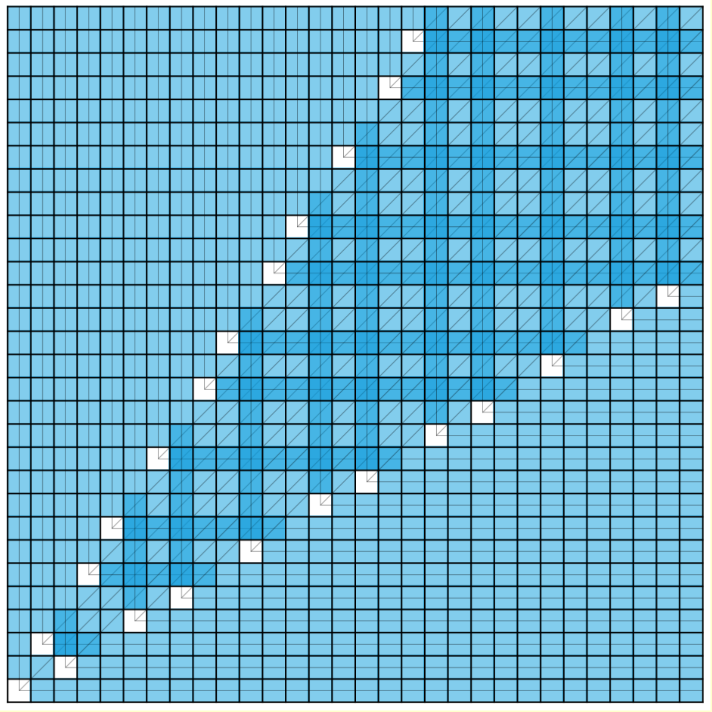


Positions which can only move to N-positions are P-positions



Wythoff's game





Wythoff sequence

(1,2) (3,5) (4,7) (6,10) (8,13) …

1. Every integer appears exactly once.

2. The *n*-th pair is different by *n*.

Wythoff sequence

n	(a_n , b_n)	a_n / n	n	(a_n , b_n)	a_n / n
1	(1,2)	1	9	(14,23)	1.5555
2	(3,5)	1.5	10	(16,26)	1.6
3	(4,7)	1.333	13	(21,34)	1.6153
4	(6,10)	1.5	34	(55,89)	1.6176
5	(8,13)	1.6	89	(144,233)	1.6179
6	(9,15)	1.5	100	(161,261)	1.61
7	(11,18)	1.571	1000	(1618,2618)	1.618
8	(12,20)	1.5	10000	(16180,26180)	1.618

Fibonacci sequence and golden ratio

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,…

Golden ratio:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887...$$

Wythoff's game

n	1	2	3	4	5	6	7
$n\varphi$	1.61	3.23	4.85	6.47	8.09	9.70	11.3
a_n	1	3	4	6	8	9	11
b_n	2	5	7	10	13	15	18

Example: Find all winning moves from (9,13)

Solution: (8,13) and (6,10)

Example 1

Find all winning moves from position (26,34).

Solution:

1. $26/1.618 \approx 16.06$, $26/2.618 \approx 9.93$

   $17 \times 1.618 \approx$ ~~27.50~~, $10 \times 2.618 \approx 26.18$

   The 10th pair is (16,26). Thus (26,16) is a winning move.

2. $34/1.618 \approx 21.01$, $34/2.618 \approx 12.98$

   $21 \times 1.618 \approx$ ~~33.97~~, $13 \times 2.618 \approx 34.03$

   The 13th pair is (21,34). Thus (21,34) is a winning move.

3. $34 - 26 = 8$

   $8 \times 1.618 \approx 12.94$, $8 \times 2.618 \approx 20.94$

Example 2

Find all winning moves from position (153,289).

Solution:

1. $153/1.618 \approx 94.56$, $153/2.618 \approx 58.44$

   $95 \times 1.618 \approx$ 153.71, $59 \times 2.618 \approx$ ~~154.46~~

   The 95th pair is (153,248). Thus (153,248) is a winning move.

2. $289/1.618 \approx 178.61$, $289/2.618 \approx 110.39$

   $179 \times 1.618 \approx$ 289.62, $111 \times 2.618 \approx$ ~~290.59~~

   The 179th pair is (289,468). No winning move for this pair.

3. $289 - 153 = 136$

   $136 \times 1.618 \approx 220.04$, $136 \times 2.618 \approx 356.04$

   The 136th pair is (220,356). No winning move for this pair.

There is one winning move: (153,248).

Wythoff's game

The n^{th} pair is

$$(a_n, b_n) = ([n\varphi], [n\varphi] + n)$$

where $[x]$ is the largest integer not larger than x. In other words, $[x]$ is the unique integer such that

$$x - 1 < [x] \leq x$$

Wythoff's game

It is easy the see that the *n*-th pair satisfies

$$b_n - a_n = n$$

To prove that every positive integer appears in the sequences exactly once, observe that

$$\frac{1}{\varphi} + \frac{1}{\varphi+1} = \frac{2}{1+\sqrt{5}} + \frac{2}{3+\sqrt{5}} = 1$$

and apply the Beatty's theorem.

Beatty's theorem

Suppose α and β are positive irrational numbers such that.

$$\frac{1}{\alpha}+\frac{1}{\beta}=1$$

Then every positive integer appears exactly once in the sequences

$$[\alpha], \quad [2\alpha], \quad [3\alpha], \quad [4\alpha], \quad [5\alpha], \cdots$$

$$[\beta], \quad [2\beta], \quad [3\beta], \quad [4\beta], \quad [5\beta], \cdots$$

Proof of Beatty's theorem

For any positive integer n, $n = [k\alpha]$ if and only if

$$k\alpha - 1 < n \le k\alpha$$

$$\Leftrightarrow k - \frac{1}{\alpha} < \frac{n}{\alpha} \le k$$

$$\Leftrightarrow k - \frac{1}{\alpha} < \left[\frac{n}{\alpha}\right] + \left\{\frac{n}{\alpha}\right\} \le k$$

where $\{x\} = x - [x]$ denotes the fractional part of x. Since α is irrational, such integer k exists if and only if

$$\left\{\frac{n}{\alpha}\right\} > 1 - \frac{1}{\alpha}$$

Proof of Beatty's theorem

We obtain, if α is a positive irrational number, then $n = [k\alpha]$ for some positive integer k if and only if

$$\left\{\frac{n}{\alpha}\right\} > 1 - \frac{1}{\alpha}$$

Proof of Beatty's theorem

Similarly, $n = [k\beta]$ for some k if and only if

$$\left\{\frac{n}{\beta}\right\} > 1 - \frac{1}{\beta}$$

Now observe that

$$\frac{n}{\alpha} + \frac{n}{\beta} = n\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = n$$

is an integer, which implies that

$$\left\{\frac{n}{\alpha}\right\} + \left\{\frac{n}{\beta}\right\} = 1 = 1 - \frac{1}{\alpha} + 1 - \frac{1}{\beta}$$

Proof of Beatty's theorem

It follows, by the irrationality of α and β again, that for any positive integer n, exactly one of

$$\left\{\frac{n}{\alpha}\right\} > 1 - \frac{1}{\alpha} \quad \text{or} \quad \left\{\frac{n}{\beta}\right\} > 1 - \frac{1}{\beta}$$

holds and therefore exactly one of the statements

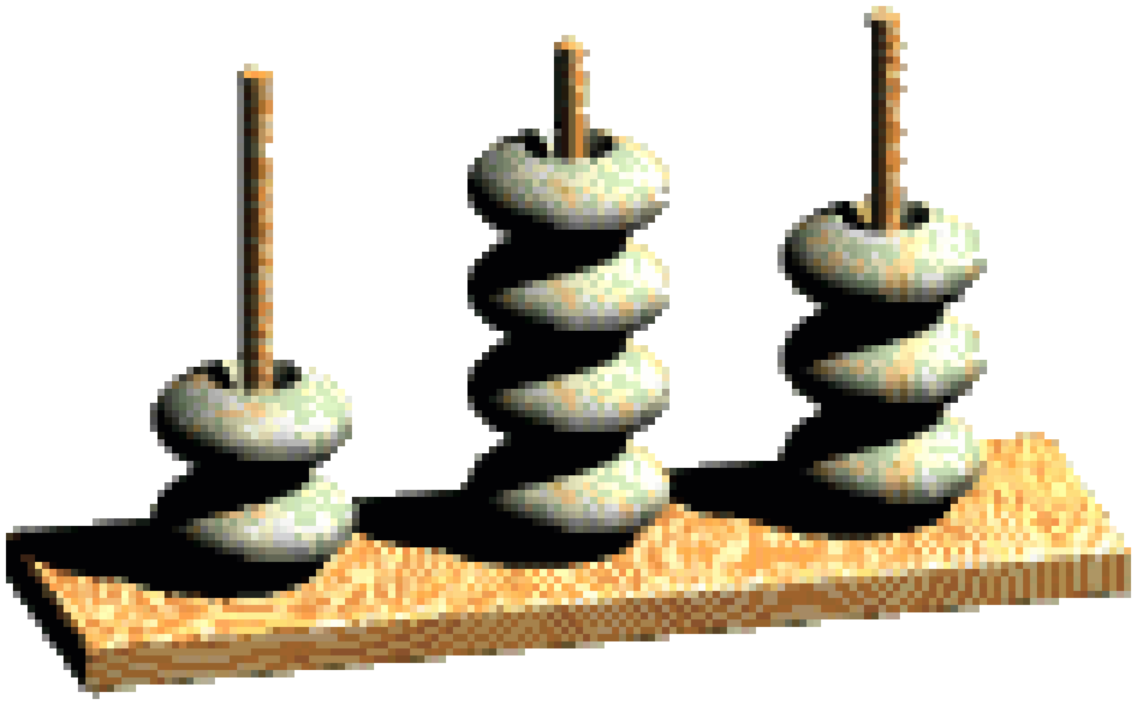
"there exists positive integer k such that $n=[k\alpha]$"

or

"there exists positive integer k such that $n=[k\beta]$"

holds and the proof of Beatty's theorem is complete.

Nim



Nim

There are three piles of chips.
On each turn , the player may

remove any number of chips
from any one of the piles.

The player who removes the last
chip wins.

Nim

We will use (x,y,z) to represent the position that there are x,y,z chips in the three piles respectively.

Nim

It is easy to see that $(x,x,0)$ is at P-position, in other words the previous player has a winning strategy. By symmetry, $(x,0,x)$ and $(0,x,x)$ are also at P-position.

Nim

By try and error one may also find the following P-positions: (1,2,3), (1,4,5), (1,6,7), (1,8,9), (2,4,6), (2,5,7), (2,8,10), (3,4,7), (3,5,6), (3,8,11),…

Nim

Binary expression:

Decimal	Binary	Decimal	Binary
1	1_2	7	111_2
2	10_2	8	1000_2
3	11_2	9	1001_2
4	100_2	10	1010_2
5	101_2	11	1011_2
6	110_2	12	1100_2

Nim

Nim-sum:
Sum of binary numbers without carry digit.

Examples:

1. $7 \oplus 5 = 2$

$$
\begin{array}{rl}
111_2 & = 7 \\
\oplus\ 101_2 & = 5 \\
\hline
10_2 & = 2
\end{array}
$$

Nim

Nim-sum:
Sum of binary numbers without carry digit.

Examples:

2. $23 \oplus 13 = 16$

$$
\begin{array}{rl}
10111_2 & = 23 \\
\oplus \quad 1101_2 & = 13 \\
\hline
11010_2 & = 26
\end{array}
$$

Nim

Properties:

1. (Associative) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
2. (Commutative) $x \oplus y = y \oplus x$
3. (Identity) $x \oplus 0 = 0 \oplus x = x$
4. (Inverse) $x \oplus x = 0$
5. (Cancellation law) $x \oplus y = x \oplus z \Rightarrow y = z$

Nim

The position (x,y,z) is at P-position if and only if

$$x \oplus y \oplus z = 0$$

Nim

P-positions:

decimal	(1,2,3)	(1,6,7)	(2,4,6)	(2,5,7)	(3,4,7)
binary	001 010 011	001 110 111	010 100 110	010 101 111	011 100 111

The number of 1’s in each column is even (either 0 or 2).

Nim

Examples:

1. (7,5,3)

$$7 \oplus 5 \oplus 3 = 1 \neq 0$$

$$\begin{array}{rl} 111_2 & = 7 \\ 101_2 & = 5 \\ \oplus \quad 11_2 & = 3 \\ \hline 1_2 & = 1 \end{array}$$

It is at N-position. Next player may win by removing 1 chip from any pile and reach P-positions (6,5,3), (7,4,3) or (7,5,2).

Nim

Examples:

2. (25,21,11)

$$25 \oplus 21 \oplus 11 = 7 \neq 0$$

It is at N-position. Next player may win by removing 3 chips from the second pile and reach P-position (25,18,11).

$$
\begin{array}{rrl}
 & 11001_2 & = 25 \\
 & 10101_2 & = 21 \\
\oplus & 1011_2 & = 11 \\
\hline
 & 111_2 & = 7
\end{array}
$$

Nim

Examples:

2. (25,21,11)

$$25 \oplus 21 \oplus 11 = 7 \neq 0$$

$$\begin{array}{rl} 11001_2 & = 25 \\ 10101_2 & = 21 \\ \oplus \quad 1011_2 & = 11 \\ \hline 111_2 & = 7 \end{array}$$

It is at N-position. Next player may win by removing 3 chips from the second pile and reach P-position $(25,18,11)$.

Note: $21 \oplus 7 = 18$