THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4250 Game Theory

Contents

1 Combinatorial games

1.1 Combinatorial games

In this chapter, we study combinatorial games.

Definition 1.1.1 (Combinatorial game)**.** *A* **combinatorial game** *is a game satisfies the following properties.*

1. *There are two players in the game.*
2. *There is a finite set of possible positions.*
3. *The players move from one position to another alternatively.*
4. *Both players know the rules and the moves of the other player. In other words, it is a game with* **perfect information***.*
5. *The game reaches a* **terminal position***, that is a position for which no further move is possible, in a finite number of moves no matter how it is played. Then one of the players is declared the winner and the other the loser.*

Note that in a combinatorial game, no random moves such as the rolling of a dice or the dealing of cards are allowed. This rules out games like backgammon and poker. A combinatorial game is a game with perfect information which does not allow simultaneous moves and hidden moves. This rules out rock-paper-scissor and battleship. Also no draws are allowed in a combinatorial game. This rules out tic-tac-to.

Given a combinatorial game, we would like to ask whether there exists a winning strategy for either one of the players. If the answer is yes, which player has a winning strategy and what is the winning strategy? First we prove the fundamental theorem of combinatorial game theory.

Theorem 1.1.2 (Zermelo's theorem)**.** *For any combinatorial game, exactly one of the players has a winning strategy.*

Proof. We prove the theorem by induction on the maximum number of moves of the game. If the maximum number of moves of a game is 1, then the theorem is obvious. Assume that the theorem is true for any game with maximum number of moves not larger than n. Suppose the maximum possible number of moves of a game G is $n+1$. Denote the first and the second player of G by I and II respectively. Let k be the number of possible choices of the first move of I. After each choice i, $i = 1, 2, \cdots, k$, of first move of I, denote the remaining game by G_i. Then the maximum number of moves of G_i is at most n for any $i = 1, 2, \cdots, k$. By induction hypothesis, for each G_i, either I or II has a winning strategy. Suppose there exists i such that I has a winning strategy for G_i. Then I has a winning strategy by choosing i in the first move. Otherwise, II has a winning strategy for every G_i which implies that II has a winning strategy for G. It is also clear that only one of the two players can have a winning strategy. □

In this notes, we only consider impartial game with normal play rule.

Definition 1.1.3. *We say that a combinatorial game is*

1. **impartial** *if the sets of moves available from any given position are the same for both players. Otherwise, it is said to be partizan.*

2. *played with* **normal play rule** *if the last player to move wins. Otherwise, it is played with* **misère play rule**.

From now on, we will assume that all combinatorial games are impartial and played with normal play rule. The following game is an example.

Example 1.1.4 (Take away game)**.** *There is a pile of n chips on a table. Two players take turns removing the chips from the table. In each turn, a player can remove either 1, 2 or 3 chips. The player removing the last chip wins.*

By Zermelo's theorem, exactly one of the players has a winning strategy. In the above take away game, it is not difficult to see that if initially there are n chips, then the first player has a winning strategy if n is not a multiple of 4 and the second player has a winning strategy if n is a multiple of 4. The winning strategy is to remove the chips so that the remaining number of chips is a multiple of 4.

1.2 P-positions and N-positions

We have seen in the last section that for any combinatorial game, exactly one of the players has a winning strategy. Thus for any position of a combinatorial game, exactly one of the players, the one who makes the previous move or the one who is going to make the next move, has a winning strategy.

Definition 1.2.1 (P-position and N-position)**.** *We say that a position of an impartial combinatorial game is a*

1. **P-position** *if the player who makes the previous move has a winning strategy.*

2. **N-position** *if the player who makes the next move has a winning strategy.*

Let us denote the player who makes the first move by I and the player who makes the second move by II. Then I has a winning strategy if the initial position is an N-position and II has a winning strategy if the initial position is a P-position. The winning strategy for the player, who has it, is always moving to a P-position.

Under the normal play rule, the player who reaches a terminal position is the winner. Thus all terminal positions are P-positions. Observe that if an N-position is reached, then the next player has a move to a position that the previous player of the position has a winning strategy. This means from an N-position, there is always a move to a P-position. On the other hand, any move from a P-position will reach a position so that the next player of the position has a winning strategy. Therefore a P-position can only move to an N-position. It turns out that these properties characterized the P-positions and N-positions.

Theorem 1.2.2 (Characterization of P-positions and N-positions)**.** *The P-positions and N-positions of an impartial game with normal play rule are determined by the following properties.*

1. *All terminal positions are P-positions.*

2. *Any move from a P-position reaches an N-position. In other words, if a position can be moved to a P-position, then it is an N-position.*

3. *Any N-position has a move to a P-position. In other words, if a position can be moved only to N-positions, then it is a P-position.*

This allows us to determine all P-positions and N-positions recursively in the following way.

Step 1. Label all terminal positions as P-positions.
Step 2. Label all positions which has a move to labeled P-position as N-positions.
Step 3. Label all positions which can be moved only to labeled N-positions as P-positions.
Step 4. If all positions are labeled, then stop; otherwise go to step 2.

After we label all positions, the winning strategy of the player is always moving to a P-position until a terminal position is reached.

For the take away game (Example 1.1.4), the set of positions is $\{0, 1, 2, 3, \cdots, n\}$. We can label the positions as follows.

1. The only terminal position is 0. We label 0 as P-position.

2. The positions which has a move to 0 are $1, 2, 3$. We label them as N-positions.

3. The position 4 can only move to $1, 2, 3$ which are N-positions. Label 4 as P position.

4. Use Step 2 to label $5, 6, 7$ as N-positions.

5. Use Step 3 to label 8 as P-position.

6. Use Step 4 to label $9, 10, 11$ as N-positions.

The above process continues until all positions are labeled. It is not difficult to see that the set of P-positions is the set of multiple of 4. To give a rigorous proof for that the multiples of 4 are exactly the P-positions, we may use the following theorem which follows directly from Theorem 1.2.2. Recall that any position is either a P-position or an N-position by Zermelo's theorem (Theorem 1.1.2). So if P is the set of P-positions of a combinatorial game, then the set of N-positions of the game is P^c, where P^c is the complement of P in the set of all positions.

Theorem 1.2.3. *A set of positions P is the set of P-positions of a combinatorial game if and only if P satisfies the following 3 properties.*

1. *All terminal positions lie in P.*

2. *For any position $p \in P$, any move from p reaches a position $q \notin P$.*

3. *For any position $q \notin P$, there exists a move from q reaching a position $p \in P$.*

Now we prove that $P = \{k : k \equiv 0 (\text{mod } 4) \text{ and } 0 \leq k \leq n\}$ is the set of P-positions of the take away game (Example 1.1.4).

Theorem 1.2.4. *The set of P-positions of the take away game (Example 1.1.4) is*

$$P = \{k : k \equiv 0 (\text{mod } 4)\}$$

Proof. We need to prove that the set P satisfy the conditions in Theorem 1.2.2.

1. The only terminal position is 0 and $0 \in P$.

2. For any position $k \in P$, any move from k will reach $k - r$, where $r = 1, 2, 3$, which is not a multiple of 4 and is not a position in P.

3. For any position $k \notin P$, k is not a multiple of 4 and let $r = 1, 2, 3$ be the remainder when k is divided by 4. Then k can be moved to $k - r \in P$.

Therefore the set P satisfies the conditions in Theorem 1.2.3 which means that P is the set of P-positions of the game. □

The following game is a generalization of the take away game.

Example 1.2.5 (Subtraction game)**.** *Let n be a positive integer and $S \subset \mathbb{Z}^+$ be a subset of the set of positive integers. The subtraction game with subtraction set S is played as follows. There is a pile of n chips where n is a position integer. Two players remove the chips in the pile alternatively. In each turn, a player removes k chips where $k \in S$. The game ends when there is no possible move and the player who makes the last move wins.*

Let's analyze the game when $S = \{1, 3, 4\}$. The labeling of P-positions and N-positions are shown below.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	⋯
P	N	P	N	N	N	N	P	N	P	N	N	N	N	P	N	⋯

One would guess that whether a position k is a P-position depends on the remainder when k is divided by 7. We may prove that it is really the case.

Theorem 1.2.6. *The set of P-positions of the subtraction game (Example 1.2.5) with* $S = \{1, 3, 4\}$ *is*

$$P = \{k : k \equiv 0, 2 (\text{mod } 7)\}$$

Proof. We need to check the following 3 conditions.

1. The only terminal position is 0 and $0 \in P$.

2. Suppose there are k chips in the pile where $k \in P$. If $k \equiv 0(\text{mod } 7)$, the positions that the next player may reach are congruent to $6, 4, 3$ modulo 7 which lie outside P. If $k \equiv 2(\text{mod } 7)$, the positions that the next player may reach are congruent to $1, 6, 5$ modulo 7 which also lie outside P. Therefore a position in P can be moved only to positions not in P.

3. Suppose there are k chips in the pile where $k \notin P$. For $k \equiv 1, 3, 4, 5, 6(\text{ mod } 7)$, the next player may remove $1, 1, 2, 3, 4$ chips and the remaining number of chips are congruent to $0, 2, 2, 2, 2$ modulo 7 respectively which are positions in P. Therefore a position not in P can always be moved to a position in P.

Therefore P is the set of P-positions of the game. □

To find the set of P-positions and N-positions of a combinatorial game, one may label the first few P-positions. Then make a guess on what the set of P-positions should be and prove that it is the case.

Example 1.2.7 (Wythoff's game)**.** *There are two piles of chips. In each turn, a player may either remove any positive number of chips from one of the piles, or remove the same positive number of chips from both piles. The player who removes the last chip wins. Determine the set of P-positions of the game.*

Solution. Use (x, y) to denote the position that there are x chips and y chips in the two piles. Note that here (x, y) and (y, x) will be considered as the same position. There is only one terminal position $(0, 0)$ and it is a P-position. Using the procedures of labeling P-positions and N-positions, one finds that the first few P-positions are as follows.

$$(0, 0), (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), \cdots$$

Denote by (a_n, b_n), $n = 0, 1, 2, 3, \cdots$, the integer pairs in the above sequence. Observe that (a_n, b_n) is uniquely determined by the following properties.

- For any $n = 0, 1, 2, 3, \cdots$, we have $b_n = a_n + n$.
- For any $0 \leq m < n$, we have $a_m < a_n$.
- For any $n = 1, 2, 3, \cdots$ and any positive integer k with $k < a_n$, there exists $m < n$ such that $a_m = k$ or $b_m = k$.

Note that every positive integer appears exactly once in the sequence of integer pairs. Now we prove

$$P = \{(a_n, b_n) : n = 0, 1, 2, \cdots\}$$

is the set of P-positions.

1. The only terminal position is $(0, 0)$ and $(0, 0) \in P$.
2. Suppose $(x, y) \notin P$ with $x \leq y$. Let $n = y - x$. Then $x \neq a_n$ otherwise $(x, y) = (a_n, a_n + n) = (a_n, b_n)$ which contradicts $(x, y) \notin P$. If $x < a_n$, then there exist $m < n$ such that $x = a_m$ or $x = b_m$. Observe that y is larger than both a_m and b_m. Thus (x, y) can be moved to $(a_m, b_m) \in P$. If $x > a_n$, then (x, y) can be moved to $(a_n, b_n) \in P$ by removing $x - a_n$ chips in both piles. Hence a position not in P has a move to a position in P.
3. Suppose $(a_n, b_n) \in P$ with $(a_n, b_n) \neq (0, 0)$. Then the next position is either $(a_n - k, b_n)$, $(a_n, b_n - k)$ or $(a_n - k, b_n - k)$ for some positive integer k. The positions $(a_n - k, b_n), (a_n, b_n - k) \notin P$ since each positive integer appears in exactly one pair. The position $(a_n - k, b_n - k) \notin P$ because $(b_n - k) - (a_n - k) = a_n - b_n = n$ and the only integer pair in P having difference equal to n is (a_n, b_n). Hence a position in P cannot reach a position in P.

Therefore the set of P-positions of the game is P. □

1.3 Nim

In this section, we study the game nim which is an important game in the study of combinatorial games.

Example 1.3.1 (Nim)**.** *There are 3 piles of chips. In each turn, a player chooses one of the 3 piles and removes any positive number of chips in the pile. The player who remove the last chip wins.*

We will use (x, y, z) to denote the position that the number of chips in the 3 piles are x, y, z. To describe the set of P-positions of nim, we first give the definition of nim-sum.

Definition 1.3.2. *Let a and b be two non-negative integers. Let $(a_n \cdots a_2a_1a_0)_2$ and $(b_n \cdots b_2b_1b_0)_2$ be the binary expressions of a and b respectively. In other words,*

$$a = 2^n a_n + \cdots + 4a_2 + 2a_1 + a_0 \text{ and } b = 2^n b_n + \cdots + 4b_2 + 2b_1 + b_0$$

Then the **nim-sum** *of a and b is defined by*

$$(a_n \cdots a_2a_1a_0)_2 \oplus (b_n \cdots b_2b_1b_0)_2 = (s_n \cdots s_2s_1s_0)_2$$

where

$$s_k = a_k + b_k (\text{mod } 2)$$

for $k = 0, 1, 2, \cdots n$. In other words, the nim-sum of a and b is the sum of binary numbers without carry digits. Nim-sum is also referred to as nimber addition.

Example 1.3.3. *Find the following nim-sum.*

1. $11 \oplus 9$
2. $25 \oplus 13$

Solution:

1. First, write the numbers in binary form $11 = 1011_2$ and $9 = 1001_2$. Now

$$\begin{array}{rrrrr} & 1 & 0 & 1 & 1_2 \\ \oplus & 1 & 0 & 0 & 1_2 \\ \hline \\ & & & 1 & 0_2 \end{array}$$

   Thus $11 \oplus 9 = 10_2 = 2$.

2. We have $25 = 11001_2$ and $13 = 1101_2$. Now

$$\begin{array}{rrrrrr} & 1 & 1 & 0 & 0 & 1_2 \\ \oplus & & 1 & 1 & 0 & 1_2 \\ \hline \\ & 1 & 0 & 1 & 0 & 0_2 \end{array}$$

   Thus $25 \oplus 13 = 10100_2 = 20$. □

The following theorem follows immediately from the definition.

Theorem 1.3.4. *Let $\mathbb{N}$ be the set of non-negative integers. The set $\mathbb{N}$ forms an Abelian group under the nim-sum $\oplus$ with 0 as identity and the inverse of an element $x \in \mathbb{N}$ is x itself. In other words,*

1. *(Associative law) For any $x, y, z \in \mathbb{N}$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.*
2. *(Identity) For any $x \in \mathbb{N}$, $x \oplus 0 = 0 \oplus x = x$.*
3. *(Inverse) For any $x \in \mathbb{N}$, $x \oplus x = 0$.*
4. *(Commutative law) For any $x, y \in \mathbb{N}$, $x \oplus y = y \oplus x$.*

A direct consequence is that $\oplus$ satisfies cancelation law.

Theorem 1.3.5 (Cancelation law)**.** *For any $x, y, z \in \mathbb{N}$, if $x \oplus z = y \oplus z$, then $x = y$.*

Proof. Suppose $x \oplus z = y \oplus z$. Then

$$x = x \oplus (z \oplus z) = (x \oplus z) \oplus z = (y \oplus z) \oplus z = y \oplus (z \oplus z) = y$$

□

There is a simple description of P-positions of the nim game in terms of nim-sum.

Theorem 1.3.6 (P-positions of nim)**.** *The set of P-positions of nim game is*

$$P = \{(x, y, z) : x \oplus y \oplus z = 0\}$$

Proof. We check the 3 conditions for P-positions as follows.

1. There is only one terminal position $(0, 0, 0)$ and $(0, 0, 0) \in P$.
2. Suppose $(x, y, z) \notin P$. Then $x \oplus y \oplus z \neq 0$. When calculating the nim-sum using the binary expressions of x, y, z, look at the leftmost column with odd number of 1. Choose a pile that the corresponding binary digit of the number of chips in the pile is 1. Remove the chips from the pile so that all columns contain even number of 1 and the nim-sum of the 3 numbers would become 0. Therefore any position not in P has a move to a position in P.

3. Suppose $(x, y, z) \in P$. Then $x \oplus y \oplus z = 0$. Suppose the position (x, y, z) is moved to (x', y', z'). Without loss of generality, we may assume that only x is changed and we have $x' < x$, $y' = y$, $z' = z$. Now if $x' \oplus y' \oplus z' = 0$, then by cancelation law,

$$x' \oplus y \oplus z = x \oplus y \oplus z \Rightarrow x' = x$$

which is a contradiction. Thus a position in P cannot move to a position in P.

Therefore P is the set of P-positions of the nim game. □

Remark: We may consider a nim game with n piles of chips for any positive integer n. The set of P-positions of the n-pile nim game is

$$P = \{(x_1, x_2, x_3, \cdots, x_n) : x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_n = 0\}$$

1.4 Sprague-Grundy function

Any combinatorial game is associated with a directed graph.

Definition 1.4.1 (Graph game)**.** *Let $G = (X, F)$ be a* **directed graph***, where X is the set of vertices and $F : X \to \mathcal{P}(X)$ is a function from X to the power set $\mathcal{P}(X)$ of X. An element $y \in F(x)$ is called a* **follower** *of x. The combinatorial game associated with directed graph G is the game with the following rules.*

1. *The set of positions is X.*
2. *A player can move from a position $x \in X$ to any follower $y \in F(x)$ of x.*
3. *A position $x \in X$ is a terminal position if $F(x) = \emptyset$.*

Conversely, for each combinatorial game, we define a directed graph associated with it as follows. The set of vertices X is the set of positions of the game. For any $x \in X$, the set of followers $F(x)$ of x is the set of positions that a player can make a move to from x. We will consider a directed graph and the combinatorial game associated to it as the same thing and denote both of them by G.

We always assume that G is **progressively bounded**. This means starting from any position $x \in X$, a terminal position must be reached in a finite number of moves. This implies in particular that $x \notin F(x)$ for any $x \in X$.

To solve the combinatorial game associated with a directed graph, we introduce the Sprague-Grundy function.

Definition 1.4.2 (Sprague-Grundy function)**.** *Let $G = (X, F)$ be a directed graph. The* **Sprague-Grundy function** *of G is the function $g : X \to \mathbb{N}$ defined by*

$$g(x) = \min\{n \geq 0 : n \neq g(y) \text{ for any } y \in F(x)\}, \text{ for } x \in X$$

The Sprague-Grundy function g is unique provided that G is progressively bounded. The value of $g(x)$ for $x \in X$ can be found recursively as follows.

1. Let $X_0 = \{x \in X : F(x) = \emptyset\}$ be the set of terminal positions. We have $g(x) = 0$ for any $x \in X_0$.

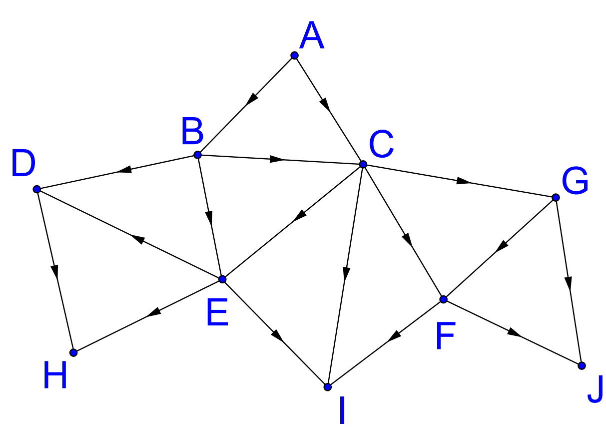
2. Suppose the value of g is known on $X_0 \cup X_1 \cup \cdots \cup X_k$. Define $X_{k+1} = \{y \in X : F(x) \subset X_0 \cup X_1 \cup \cdots \cup X_k\}$. For any $x \in X_{k+1}$, we have

$$g(x) = \min\{n \geq 0 : n \neq g(y) \text{ for any } y \in F(x)\}$$

   Note that for any $y \in F(x)$, $g(y)$ is known because $y \in X_0 \cup X_1 \cup \cdots \cup X_k$.

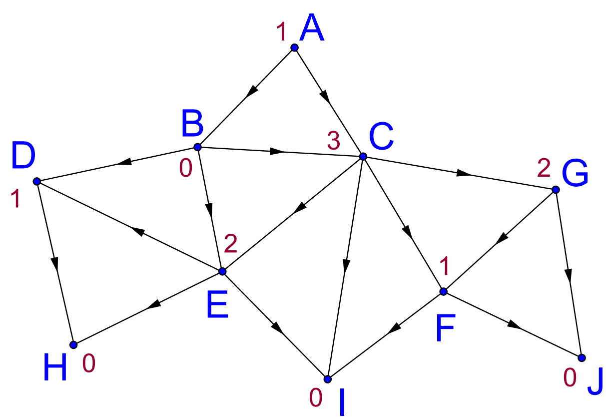
3. Repeat step 2 until $g(x)$ is known for any $x \in X$.

Example 1.4.3. *Consider the directed graph*



The set of vertices is $X = \{A, B, C, D, E, F, G, H, I, J\}$. *We may label the values of the Sprague-Grundy function of the vertices with the following steps.*

1. *Label terminal positions* H, I, J *as 0*
2. *Label* D, F *as 1*
3. *Label* E, G *as 2*
4. *Label* B *as 0 and* C *as 3*
5. *Label* A *as 1*



The follower function and the values of Sprague-Grundy function are shown in the following table.

x	$F(x)$	$g(x)$
A	$\{B, C\}$	1
B	$\{C, D, E\}$	0
C	$\{E, F, G, I\}$	3
D	$\{H\}$	1
E	$\{D, H, I\}$	2
F	$\{I, J\}$	1
G	$\{F, J\}$	2
H	$\emptyset$	0
I	$\emptyset$	0
J	$\emptyset$	0

□

The procedures described above can be used to guess a formula for the Sprague-Grundy function of a game. To prove that a function $g(x)$ is the Sprague-Grundy function of a game $G = (X, F)$, we need to show the following two statements.

1. For any $x \in X$, if k is a non-negative integer such that $k < g(x)$, then there exists $x' \in F(x)$ such that $g(x') = k$.

2. For any $x \in X$ and $x' \in F(x)$, we have $g(x') \neq g(x)$.

Example 1.4.4 (Sprague-Grundy function of subtraction game)**.** *The Sprague-Grundy function of the subtraction game $G(m)$ with subtraction set $S = \{1, 2, \cdots, m\}$ (Example 1.2.5) where $m \in \mathbb{Z}^+$ is the function $g : \mathbb{N} \to \{0, 1, 2, \cdots, m\}$ defined by $g(x) \equiv x (\text{mod } m + 1)$.*

Proof. We need to prove that g satisfies the following two conditions.

1. For any $0 \leq k < g(x)$, there exists $x' \in F(x)$ such that $g(x') = k$:

   For any $0 \leq k < g(x)$, we have $0 < g(x) - k \leq m$. Thus $x' = x - (g(x) - k) \in F(x)$ which corresponds removing $g(x) - k$ chips from the pile. Then we have $g(x') = g(x - g(x) + k) = k$ since $x - g(x)$ is a multiple of $m + 1$.

2. For any $x' \in F(x)$, we have $g(x') \neq g(x)$:

   For any $x' \in F(x)$, we have $x - x' = 1, 2, \cdots, m$ which implies $x' \not\equiv x (\text{mod } m + 1)$. Thus $g(x') \neq g(x)$.

This completes the proof that g is the Sprague-Grundy function of G. □

Example 1.4.5. *Let $X = \{0, 1, 2, \cdots, n\}$ where n is a positive integer. Let $S \subset \mathbb{Z}^+$ be a collection of positive integers. For any $x \in X$, define $F(x) = \{y \in X : y = x - s \text{ for some } s \in S\}$. Then the game associated with $G = (X, F)$ is the subtraction game with subtraction set S (Example 1.2.5). Now suppose $S = \{1, 3, 4\}$. We list the values of g below*

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(x)$	0	1	0	1	2	3	2	0	1	0	1	2	3

In fact, it is not difficult to see that

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0, 2 \pmod 7 \\ 1, & \text{if } x \equiv 1, 3 \pmod 7 \\ 2, & \text{if } x \equiv 4, 6 \pmod 7 \\ 3, & \text{if } x \equiv 5 \pmod 7 \end{cases}$$

□

Example 1.4.6. *The values of the Sprague-Grundy function $g(x, y)$ of the Wythoff's game (Example 1.2.7) are listed below.*

$x \backslash y$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	0	4	5	3	7	8
2	2	0	1	5	3	4	8	6
3	3	4	5	6	2	0	1	9
4	4	5	3	2	7	6	9	0
5	5	3	4	0	6	8	10	1
6	6	7	8	1	9	10	3	4
7	7	8	6	9	0	1	4	5

□

Example 1.4.7 (At least half game)**.** *There are n chips on the table. On each turn, a player may remove at least half of the chips. The values of the Sprague-Grundy function of the game are listed below.*

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(x)$	0	1	2	2	3	3	3	3	4	4	4	4	4

More precisely

$$g(x) = \min\{k : 2^k > x\}$$

□

The at least half game is a rather silly game. One can win at the first move by removing all the chips. In other words, the only P-position of the game is the terminal position 0. However if the game is played with several piles in stead of just one, the Sprague-Grundy function will be helpful in finding a winning strategy.

The following theorem shows how to find the set of P-positions using the Sprague-Grundy function.

Theorem 1.4.8. *Let $G = (X, F)$ be a directed graph and g be the Sprague-Grundy function of G. Then the set of P-positions of the combinatorial game associated with G is*

$$P = \{x \in X : g(x) = 0\}$$

Proof. We prove that the set P satisfies the conditions for P-positions (Theorem 1.2.3).

1. For any terminal position x, we have $F(x) = \emptyset$. Then $\{n \geq 0 : n \neq g(y) \text{ for any } y \in F(x)\} = \mathbb{N}$ which implies $g(x) = 0$. Thus $x \in P$.

2. For any $x \notin P$, we have $g(x) \neq 0$. Then there exists $y \in F(x)$ such that $g(y) = 0$ which means $y \in P$. Thus any position not in P has a follower in P.

3. For any $x \in P$, we have $g(x) = 0$. Then for any $y \in F(x)$, we must have $g(y) \neq 0$ for otherwise $g(x)$ cannot be 0. Thus any follower of a position in P does not lie in P.

Therefore P is the set of P-positions of the game. □

To find the Sprague-Grundy function $g(x_1, x_2, x_3)$ of nim game, we know that g takes the value 0 when (x_1, x_2, x_3) is a P-position which means that $x_1 \oplus x_2 \oplus x_3 = 0$. It is then very natural to guess that $g(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$. The following theorem says that it is in fact the case.

Theorem 1.4.9. *Let $g(x_1, x_2, x_3)$ be the Sprague-Grundy function of nim game (Example 1.3.1). Then*

$$g(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$$

for any position (x_1, x_2, x_3) where $\oplus$ denotes the nim-sum.

Proof. Let $x = (x_1, x_2, x_3)$ be a position. To prove that $g(x) = x_1 \oplus x_2 \oplus x_3$ is the Sprague-Grundy function, we need to prove two statements.

1. For any $0 \leq m < g(x)$, there exists $y \in F(x)$ such that $g(y) = m$:

   Consider the leftmost digit of $m \oplus g(x)$. The digit of $g(x) = x_1 \oplus x_2 \oplus x_3$ at this digit is 1 since $m < g(x)$. Then at least one of the digits of x_1, x_2, x_3 at this digit is 1. Without loss of generality, we may assume that the digit of x_1 at this digit is 1. It follows that $m \oplus g(x) \oplus x_1 < x_1$. Let $y_1 = m \oplus g(x) \oplus x_1$. Then the position $y = (y_1, x_2, x_3)$ is a follower of x and we have

$$\begin{aligned} g(y) &= y_1 \oplus x_2 \oplus x_3 \\ &= m \oplus g(x) \oplus x_1 \oplus x_2 \oplus x_3 \\ &= m \oplus g(x) \oplus g(x) \\ &= m \end{aligned}$$

2. For any $y \in F(x)$, we have $g(y) \neq g(x)$:

   Without loss of generality, we may assume that $y = (y_1, x_2, x_3)$ with $y_1 < x_1$. Now if $g(y) = g(x)$, then $y_1 \oplus x_2 \oplus x_3 = x_1 \oplus x_2 \oplus x_3$. This implies $y_1 = x_1$ by cancelation law of nim-sum which contradicts $y_1 < x_1$. Therefore we must have $g(y) \neq g(x)$.

This completes the proof that g is the Sprague-Grundy function of the nim game. □

1.5 Sum of combinatorial games

Suppose we have n combinatorial games $G_1, G_2, \cdots, G_n$. We may play a new game G with the following rules.

1. On each turn, a player selects one of the games and makes a move in that game leaving all other games untouched.

2. The game G is at a terminal position if all games are at terminal positions.

The game described above is call the sum of the games $G_1, G_2, \cdots, G_n$ and is denoted by $G = G_1 + G_2 + \cdots + G_n$. As we have seen in the last section, each combinatorial game is associated with a directed graph. The sum of games can be defined using the language of directed graph as follows. Noted that we consider a directed graph and the combinatorial game associated with it as the same thing.

Definition 1.5.1 (Sum of combinatorial games)**.** *The* **sum of the combinatorial games** $G_1 = (X_1, F_1)$, $G_2 = (X_2, F_2)$, $\cdots, G_n = (X_n, F_n)$ *is defined as* $G = (X, F)$ *where*

$$X = X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) : x_k \in X_k\}$$

and $F : X \to X$ *is defined by*

$$\begin{aligned} & F(x_1, x_2, \cdots, x_n) \\ = & \bigcup_{k=1}^{n} \{x_1\} \times \cdots \{x_{k-1}\} \times F_k(x_k) \times \{x_{k+1}\} \times \cdots \times \{x_n\} \\ = & F_1(x_1) \times \{x_2\} \times \cdots \times \{x_n\} \bigcup \{x_1\} \times F(x_2) \times \cdots \times \{x_n\} \\ & \bigcup \cdots \bigcup \{x_1\} \times \cdots \times \{x_{n-1}\} \times F_n(x_n) \end{aligned}$$

Theorem 1.5.2 (Sprague-Grundy theorem)**.** *Let* $g_1, g_2, \cdots, g_n$ *be the Sprague-Grundy functions of the combinatorial games* $G_1 = (X_1, F_1)$, $G_2 = (X_2, F_2)$, $\cdots, G_n = (X_n, F_n)$ *respectively. Then the Sprague-Grundy function of the sum* $G = G_1 + G_2 + \cdots + G_n$ *is*

$$g(x_1, x_2, \cdots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

Proof. Let $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$. To prove that g is the Sprague-Grundy function of G, we need to prove two statements.

1. For any $0 \leq m < g(x)$, there exists $x' \in F(x)$ such that $g(x') = m$:

   For any $0 \leq m < g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$, consider the leftmost digit of $m \oplus g(x)$. The digit of $g(x)$ at this digit is 1 since $m < g(x)$ and there exists $1 \leq k \leq n$ such that the digit of $g_k(x_k)$ at this digit is 1. It follows that $m \oplus g(x) \oplus g_k(x_k) < g(x_k)$ and there exists $x'_k \in F_k(x_k)$ such that $g_k(x'_k) = m \oplus g(x) \oplus g_k(x_k)$. Now take

$x' = (x_1, \cdots, x_{k-1}, x'_k, x_{k+1}, \cdots, x_n)$. Then we have $\in F(x)$ and

$$\begin{aligned} & g(x') \\ = \; & g_1(x_1) \oplus \cdots \oplus g_{k-1}(x_{k-1}) \oplus g_k(x'_k) \oplus g_{k+1}(x_{k+1}) \oplus \cdots \oplus g_n(x_n) \\ = \; & g(x) \oplus g_k(x'_k) \oplus g_k(x_k) \qquad \text{(Note: } x \oplus x = 0 \text{ for any } x.) \\ = \; & g(x) \oplus (m \oplus g(x) \oplus g_k(x_k)) \oplus g_k(x_k) \\ = \; & m \end{aligned}$$

2. For any $x' \in F(x)$, we have $g(x') \neq g(x)$:

   We prove the statement by contradiction. Suppose there exists $x' = (x_1, \cdots, x_{k-1}, x'_k, x_{k+1}, \cdots, x_n) \in F(x)$, where x'_k is a follower of x_k in G_k, such that $g(x') = g(x)$. Then $g_k(x'_k) = g_k(x_k)$ by cancelation law (Theorem 1.3.5) which is impossible because g_k is the Sprague-Grundy function of G_k.

Therefore g is the Sprague-Grundy function of G. □

Example 1.5.3 (n-pile nim game)**.** *In the 1-pile nim game, there is one pile of chips and in each turn, a player can remove any positive number of chips in the pile. Denote the 1-pile nim game by G. The n-pile nim game G_n is the sum of n copies of the 1-pile nim game G. Since the Sprague-Grundy function of G is $g(x) = x$, the Sprague-Grundy function of G_n is*

$$\begin{aligned} g_n(x_1, x_2, \cdots, x_n) &= g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(x_n) \\ &= x_1 \oplus x_2 \oplus \cdots \oplus x_n \end{aligned}$$

When $n = 1$, the Sprague-Grundy function of $G_1 = G$ is $g(x) = x$. The only P-position is the terminal position $x = 0$. The winning strategy is to remove all chips from the pile in the first step.

When $n = 2$, the Sprague-Grundy function of $G_2 = G + G$ is $g(x_1, x_2) = x_1 \oplus x_2$. The set of P-positions of G_2 is

$$\begin{aligned} P &= \{(x_1, x_2) : x_1 \oplus x_2 = 0\} \\ &= \{(x_1, x_2) : x_1 = x_2\} \end{aligned}$$

Thus a position of G_2 is a P-position if and only if the number of chips in the two piles are the same.

When $n = 3$, G_3 is the ordinary nim game and its Sprague-Grundy function is $g(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$. □

Example 1.5.4. *Let $G(m)$ be the subtraction game (Example 1.2.5) with subtraction set $S = \{1, 2, \cdots, m\}$. Recall that (Theorem 1.4.4) the Sprague-Grundy function of $G(m)$ is $g_m : \mathbb{N} \to \{0, 1, 2, \cdots, m\}$ defined by $g_m(x) = x(\text{mod } m + 1)$. Consider the game $G = G(3) + G(6) + g(9)$. For $x = (10, 13, 17)$, we have $g(x) = 2 \oplus 6 \oplus 7 = 3 \neq 0$. Thus $(10, 13, 17)$ is an N-position. To win the game, the next player can make a move to either $(9, 13, 17)$, $(10, 12, 17)$ or $(10, 13, 14)$. The reader may check that these are P-positions, in other words with value of g equals to 0.*

Example 1.5.5. *Consider the sum of the following 3 games:*

Game 1: The 1-pile nim game (Example 1.5.3).

Game 2: The at least half game (Example 1.4.7).

Game 3: The subtraction game $G(15)$ with subtraction set $\{1, 2, \cdots, 15\}$ (Example 1.4.4).

The Sprague-Grundy function of Game 1, Game 2, Game 2 are

$$\begin{aligned} g_1(x) &= x \\ g_2(x) &= \min\{k : 2^k > x\} \\ g_3(x) &= x(\text{mod } 16) \end{aligned}$$

respectively. The position $(15, 19, 28)$ has Sprague-Grundy value

$$\begin{aligned} g(15, 19, 28) &= g_1(15) \oplus g_2(19) \oplus g_3(28) \\ &= 15 \oplus 5 \oplus 12 \\ &= 6 \end{aligned}$$

Therefore $(15, 19, 28)$ is an N-position. Now the position $(15, 5, 12)$ of the nim game has winning moves to $(9, 5, 12)$, $(15, 3, 12)$ and $(15, 5, 10)$. A P-position is reached if the Sprague-Grundy values of the three games are one of the above triple. The possible moves are listed in the following table.

Game	*original position*	*original g value*	*final g value*	*possible moves*
1-pile nim	*15*	*15*	*9*	*9*
at least half	*19*	*5*	*3*	*4,5,6,7*
$G(15)$	*28*	*12*	*10*	*26*

Hence P-positions that the next player can reach are $(9, 19, 28)$, $(15, 4, 28)$, $(15, 5, 28)$, $(15, 6, 28)$, $(15, 7, 28)$ *and* $(15, 19, 26)$. □

Example 1.5.6 (Take or break game)**.** *There are several piles of chips. In each turn, a player may either*

1. *remove any positive number of chips from one of the piles, or*

2. *split one of the piles into two non-empty piles.*

The values of the Sprague-Grundy function of the game are obtained from the following table.

x	*possible splitting*	*Sprague-Grundy values of possible splitting*	$g(x)$
0	*none*		*0*
1	*none*		*1*
2	$(1,1)$	*0*	*2*
3	$(1,2)$	*3*	*4*
4	$(1,3),(2,2)$	*5,0*	*3*
5	$(1,4),(2,3)$	*2,6*	*5*
6	$(1,5),(2,4),(3,3)$	*4,1,0*	*6*
7	$(1,6),(2,5),(3,4)$	*7,7,7*	*8*
8	$(1,7),(2,6),(3,5),(4,4)$	*9,4,1,0*	*7*
9	$(1,8),(2,7),(3,6),(4,5)$	*6,10,2,6*	*9*

It can be proved that the Sprague-Grundy function of the game is given by

$$g(x) = \begin{cases} 4k+1, & \text{if } x = 4k+1 \\ 4k+2, & \text{if } x = 4k+2 \\ 4k+4, & \text{if } x = 4k+3 \\ 4k+3, & \text{if } x = 4k+4 \end{cases}$$

Consider the position $(3, 5, 8)$*. Its Sprague-Grundy value is*

$$\begin{aligned} g(3,5,8) &= g(3) \oplus g(5) \oplus g(8) \\ &= 4 \oplus 5 \oplus 7 \\ &= 6 \end{aligned}$$

Thus $(3,5,8)$ *is an N-position. We may find the winning moves by looking at the nim-sum*

$$\begin{array}{cccc} & 1 & 0 & 0_2 \\ & 1 & 0 & 1_2 \\ \oplus & 1 & 1 & 1_2 \\ \hline & 1 & 1 & 0_2 \end{array}$$

The winning moves and the corresponding Sprague-Grundy values are listed in the following table.

Winning move	*Sprague-Grundy value*
$(2,5,8)$	$g(2,5,8) = 2 \oplus 5 \oplus 7 = 0$
$(3,4,8)$	$g(3,4,8) = 4 \oplus 3 \oplus 7 = 0$
$(3,5,1)$	$g(3,5,1) = 4 \oplus 5 \oplus 1 = 0$
$(3,5,3,5)$	$g(3,5,3,5) = 4 \oplus 5 \oplus 4 \oplus 5 = 0$

Sometimes we may get a winning move by increasing the Sprague-Grundy value. Consider the position $(1,7,8)$. *The Sprague-Grundy value is*

$$\begin{aligned} g(1,7,8) &= g(1) \oplus g(7) \oplus g(8) \\ &= 1 \oplus 8 \oplus 7 \\ &= 14 \end{aligned}$$

Consider the nim-sum

$$\begin{array}{ccccc} & 0 & 0 & 0 & 1_2 \\ & 1 & 0 & 0 & 0_2 \\ \oplus & 0 & 1 & 1 & 1_2 \\ \hline & 1 & 1 & 1 & 0_2 \end{array}$$

We see that a winning move is $(1,6,8)$. *However there is one more winning move by increasing the Sprague-Grundy value of 8. This can be done by splitting* 8 *into* $(1,7)$. *The winning moves are listed in the following table.*

Winning move	*Sprague-Grundy value*
$(1,6,8)$	$g(1,6,8) = 1 \oplus 6 \oplus 7 = 0$
$(1,7,1,7)$	$g(1,7,1,7) = 1 \oplus 8 \oplus 1 \oplus 8 = 0$

□

Suppose we have n combinatorial games $G_1, G_2, \cdots, G_n$ and $G = G_1 + G_2 + \cdots + G_n$ is their sum. One remarkable consequence of the Sprague-Grundy theorem (Theorem 1.5.2) is that G is some how equivalent to the n-pile nim game. The position $(x_1, x_2, \cdots, x_n)$ of G corresponds to the position $(g_1(x_1), g_2(x_2), \cdots, g_n(x_n))$ of the n-pile nim game where $g_1, g_2, \cdots, g_n$ are Sprague-Grundy functions of $G_1, G_2, \cdots, G_n$ respectively. Suppose $(x_1, x_2, \cdots, x_n)$ is an N-position. The next player has a winning strategy as follows. Since $g(x_1, x_2, \cdots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) \neq 0$ which means $(g_1(x_1), g_2(x_2), \cdots, g_n(x_n))$ is an N-position of the nim game. Consequently, the next player has a winning strategy by removing chips from the k-th pile leaving the other $n-1$ piles unchange to reach $(g_1(x_1), \cdots, g_{k-1}(x_{k-1}), m, g_{k+1}(x_{k+1}) \cdots, g_n(x_n))$ for some $m < g_k(x_k)$. Now there exists x'_k which is a follower of x_k such that $g_k(x'_k) = m$. Then the next player can win by moving from $(x_1, x_2, \cdots, x_n)$ to $(x_1, \cdots, x_{k-1}, x'_k, x_{k+1}, \cdots, x_n)$ which is a P-position.

Exercise 1

1. Let $\oplus$ denotes the nim-sum.
   (a) Find $27 \oplus 17$
   (b) Find x if $x \oplus 38 = 25$.
   (c) Prove that if $x \oplus y \oplus z = 0$, then $x = y \oplus z$.
2. Let $\oplus$ denotes the nim-sum.
   (a) Find $29 \oplus 20 \oplus 15$.
   (b) Find all winning moves of the game of nim from the position $(29, 20, 15)$.
3. Find all winning moves in the game of nim,
   (a) with three piles of 12, 19, and 27 chips.
   (b) with four piles of 13, 17, 19, and 23 chips.
4. Consider the subtraction game with $S = \{1, 3, 4, 5\}$.
   (a) Find the set of P-positions of the game.
   (b) Prove your assertion in (a).

(c) Let $g(x)$ be the Sprague-Grundy function of the game. Find $g(4)$, $g(18)$ and $g(29)$.

5. Let $g(x)$ be the Sprague-Grundy function of the subtraction game with subtraction set $S = \{1, 2, 6\}$.

   (a) Find $g(4)$, $g(6)$ and $g(100)$.

   (b) Find all winning moves for the first player if initially there are 100 chips.

   (c) Find the set of P-positions of the game and prove your assertion.

6. In a 2-pile take-away game, there are 2 piles of chips. In each turn, a player may either remove any number of chips from one of the piles, or remove the same number of chips from both piles. The player removing the last chip wins.

   (a) Find all winning moves for the starting positions $(6, 9)$, $(11, 15)$ and $(13, 20)$.

   (b) Find (x, y) if (x, y) is a P-position and

      (i) $x = 100$

      (ii) $x = 500$

      (iii) $x - y = 999$

7. In a staircase nim game there are 5 piles of coins. Two players take turns moving. A move consists of removing any number of coins from the first pile or moving any number of coins from the k-th pile to the $k-1$-th pile for $k = 2, 3, 4, 5$. The player who takes the last coin wins. Let $(x_1, x_2, \cdots, x_5)$ denotes the position with x_i coins in the i-th pile.

   (a) Prove that $(x_1, x_2, \cdots, x_5)$ is a P-position if and only if (x_1, x_3, x_5) is a P-position in the ordinary nim.

   (b) Determine all winning moves from the initial position $(4, 6, 9, 11, 14)$.

8. Consider the following 3 games with normal play rule.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6\}$
Game 3:	When there are n chips remaining, a player can only remove 1 chip if n is odd and can remove any even number of chips if n is even.

Let g_1, g_2, g_3 be the the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the 3 games and g be the Sprague-Grundy function of G.

(a) Find $g_1(14), g_2(17), g_3(24)$.

(b) Find $g(14, 17, 24)$.

(c) Find all winning moves of G from the position $(14, 17, 24)$.

9. Consider the following 3 games.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6, 7\}$
Game 3:	When there are n chips remaining, a player can remove any **odd** number of chips if n is odd and can remove 1 or 2 chips if n is even.

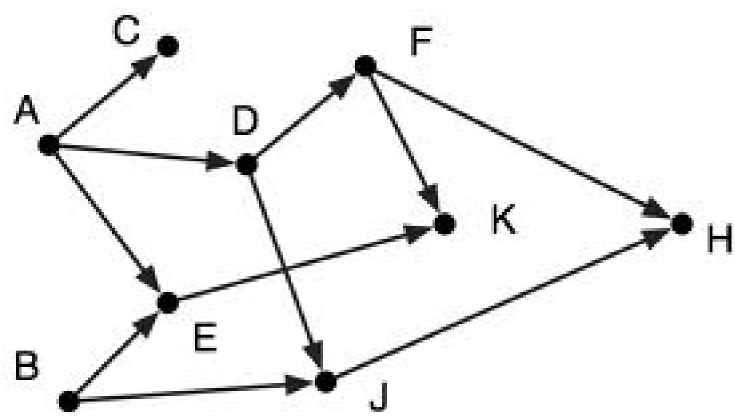
Let g_1, g_2, g_3 be the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.

(a) Find $g_1(7), g_2(14), g_3(18)$.

(b) Find $g(7, 14, 18)$.

(c) Find all winning moves of G from the position $(7, 14, 18)$.

10. Consider the game associated with the following directed graph



(a) Copy the graph and write down the value of the Sprague-Grundy function of each vertex.

(b) Write down all vertices which are at P-position but not at terminal position.

(c) Consider the sum of three copies of the given graph game.

  (i) Find $g(A, B, E)$ where g is the Sprague-Grundy function.

  (ii) Find all winning moves from (A, B, E).

11. Let $g(x)$ be the Sprague-Grundy function of the take-and-break game.

  (a) Write down $g(10)$, $g(11)$, $g(12)$.

  (b) Find all winning moves from $(10, 11, 12)$

2 Two-person zero sum games

2.1 Game matrices

In a two-person zero sum game, two players, player I and player II, make their moves simultaneously. Then the payoffs to the players depend on the strategies used by the players. In this chapter, we study only **zero sum games** which means the sum of the payoffs to the players is always zero. We will also assume that the game has **perfect information** which means all players know how the outcomes depend on the strategies the players use.

Definition 2.1.1 (Strategic form of a two-person zero sum game)**.** *The* **strategic form** *of a two-person zero sum game is given by a triple* (X, Y, π) *where*

1. X *is the set of strategies of player* I*.*
2. Y *is the set of strategies of player* II*.*
3. $\pi : X \times Y \to \mathbb{R}$ *is the payoff function of player* I*.*

For $(x, y) \in X \times Y$, the value $\pi(x, y)$ is the payoff to player I when player I uses strategy x and player II uses strategy y. Note that the payoff to player II is equal to $-\pi(x, y)$ since the game is a zero sum game. The game has perfect information means that the function π is known to both players. We will always assume that the sets X and Y are finite. In this case we may assume $X = \{1, 2, \cdots, m\}$ and $Y = \{1, 2, \cdots, n\}$. Then the payoff function can be represented by an $m \times n$ matrix which is called the **game matrix** and we will denote it by $A = [a_{ij}]$. A two-person zero sum game is completely determined by its game matrix. When player I uses the i-th strategy and player II uses the j-th strategy, then the payoff to player I is the entry a_{ij} of A. The payoff to player II is then $-a_{ij}$. If a two-person zero sum game is represented by a game matrix, we will call player I the **row player** and player II the **column player**.

Given a game matrix A, we would like to know what the optimal strategies for the players are and what the payoffs to the players will be if both of them use their optimal strategies. The answer to this question is simple if A has a saddle point.

Definition 2.1.2 (Saddle point)**.** *We say that an entry* a_{kl} *is a* **saddle point** *of an* $m \times n$ *matrix* A *if*

1. $a_{kl} = \min_{j=1,2,\cdots,n}\{a_{kj}\}$
2. $a_{kl} = \max_{i=1,2,\cdots,m}\{a_{il}\}$

The first condition means that when the row player uses the k-th strategy, then the payoff to the row player is not less than a_{kl} no matter how the column player plays. The second condition means that when the column player uses the l-th strategy, then the payoff to the row player is not larger than a_{kl} no matter how the row player plays. Consequently we have

Theorem 2.1.3. *If A has a saddle point a_{kl}, then the row player may guarantee that his payoff is not less than a_{kl} by using the k-th strategy and the column player may guarantee that the payoff to the row player is not larger than a_{kl} by using the l-th strategy.*

Suppose A is a matrix which has a saddle point a_{kl}. The above theorem implies that the corresponding row and column constitute the optimal strategies for the players. To find the saddle points of a matrix, first write down the row minima of the rows and the column maxima of the columns. Then find the maximum of row minima which is called the **maximin**, and the minimum of the column maxima which is called the **minimax**. If the maximin is equal to the minimax, then the entry in the corresponding row and column is a saddle point. If the maximin and minimax are different, then the matrix has no saddle point.

Example 2.1.4.

$$\begin{array}{cc} & \begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & 2 \\ 0 & -4 & -3 \\ -2 & 4 & 1 \end{pmatrix} \begin{array}{c} \textit{min} \\ 0 \\ 2 \\ -4 \\ -2 \end{array} \\ \textit{max} & \begin{array}{ccc} 3 & 5 & 2 \end{array} \end{array}$$

Both the maximin and minimax are 2. Therefore the entry $a_{23} = 2$ is a saddle point. □

Example 2.1.5.

$$\begin{array}{ccccccc} & & & & & & \textit{min} \\ & \left(\begin{array}{cccc} 2 & -1 & 3 & 1 \\ -4 & 2 & 0 & 3 \\ 0 & 1 & -2 & 4 \end{array}\right) & & & & & \begin{array}{c} -1 \\ -4 \\ -2 \end{array} \\ \textit{max} & \begin{array}{cccc} 2 & 2 & 3 & 4 \end{array} \end{array}$$

The maximin is -1 *while the minimax is* 2 *which are not equal. Therefore the matrix has no saddle point.* □

Saddle point of a matrix may not be unique. However the values of saddle points are always the same.

Theorem 2.1.6. *The values of the saddle points of a matrix are the same. That is to say, if* a_{kl} *and* a_{pq} *are saddle points of a matrix, then* $a_{kl} = a_{pq}$*. Furthermore, we have* $a_{pq} = a_{pl} = a_{kq} = a_{kl}$*.*

Proof. We have

$$\begin{aligned} a_{kl} &\leq a_{kq} \quad (\text{since } a_{kl} \leq a_{kj} \text{ for any } j) \\ &\leq a_{pq} \quad (\text{since } a_{iq} \leq a_{pq} \text{ for any } i) \\ &\leq a_{pl} \quad (\text{since } a_{pq} \leq a_{pj} \text{ for any } j) \\ &\leq a_{kl} \quad (\text{since } a_{il} \leq a_{kl} \text{ for any } i) \end{aligned}$$

Therefore

$$a_{kl} = a_{kq} = a_{pq} = a_{pl}$$

□

We have seen that if A has a saddle point, then the two players have optimal strategies by using one of their strategies constantly (Theorem 2.1.3). If A has no saddle point, it is expected that the optimal ways for the players to play the game are not using one of the strategies constantly. Take the rock-paper-scissors game as an example.

Example 2.1.7 (Rock-paper-scissors)**.** *The* **rock-paper-scissors** *game has the game matrix*

$$\begin{array}{cc} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \left(\begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array}\right) \end{array}$$

Here we use the order rock(R), paper(P), scissors(S) to write down the game matrix. □

Everybody knows that the optimal strategy of playing the rock-paper-scissors game is not using any one of the gestures constantly. When one of the strategies of a player is used constantly, we say that it is a **pure strategy**. For games without saddle point like rock-paper-scissors game, mixed strategies instead of pure strategies should be used.

Definition 2.1.8 (Mixed strategy). *A* **mixed strategy** *is a row vector* $\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ *such that*

1. $x_i \geq 0$ *for any* $i = 1, 2, \cdots, m$
2. $\sum_{i=1}^{m} x_i = 1$

In other words, a vector is a mixed strategy if it is a **probability vector**. *We will denote the set of probability* m *vectors by* $\mathcal{P}^m$.

When a mixed strategy $(x_1, x_2, \cdots, x_m)$ is used, the player uses his i-th strategy with a probability of x_i for $i = 1, 2, \cdots, m$. Mixed strategies are generalization of pure strategies. If one of the coordinates of a mixed strategy is 1 and all other coordinates are 0, then it is a pure strategy. So a pure strategy is also a mixed strategy. Suppose the row player and the column player use mixed strategies $\mathbf{x} \in \mathcal{P}^m$ and $\mathbf{y} \in \mathcal{P}^n$ respectively. Then the outcome of the game is not fixed because the payoffs to the players will then be random variables. We denote by $\pi(\mathbf{x}, \mathbf{y})$ the **expected payoff** to the row player when the row player uses mixed strategy $\mathbf{x}$ and the column player uses mixed strategy $\mathbf{y}$. We have the following simple formula for the expected payoff $\pi(\mathbf{x}, \mathbf{y})$ to the row player.

Theorem 2.1.9. *In a two-person zero sum game with* $m \times n$ *game matrix* A*, suppose the row player uses mixed strategies* $\mathbf{x}$ *and the column player uses mixed strategies* $\mathbf{y}$ *independently. Then the expected payoff to the row player is*

$$\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x} A \mathbf{y}^T$$

where $\mathbf{y}^T$ *is the transpose of* $\mathbf{y}$.

Proof. The expected payoff to the row player is

$$\begin{aligned}
& E(\text{payoff to the row player}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy and } II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy}) P(II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} x_i y_j \\
= & \mathbf{x} A \mathbf{y}^T
\end{aligned}$$

□

Let A be an $m \times n$ game matrix. For $\mathbf{x} \in \mathcal{P}^m$, the vector

$$\mathbf{x}A \in \mathbb{R}^n$$

has the following interpretation. The j-th coordinate, $j = 1, 2, \cdots, n$, of the vector is the expected payoff to the row player if the row player uses mixed strategy $\mathbf{x}$ and the column player uses the j-th strategy constantly. In this case a rational column player would use the l-th strategy, $1 \le l \le n$, such that the l-th coordinate of the vector $\mathbf{x}A$ is the least coordinate among all coordinates of $\mathbf{x}A$. (Note that the column player wants the expected payoff to the row player as small as possible since the game is a zero sum game.)

On the other hand, for $\mathbf{y} \in \mathcal{P}^n$, the i-th coordinate, $i = 1, 2, \cdots, m$, of the column vector

$$A\mathbf{y}^T \in \mathbb{R}^m$$

is the expected payoff to the row player if the row player uses his i-th strategy constantly and the column player uses the mixed strategy $\mathbf{y}$. In this case a rational row player would use the k-th strategy, $1 \le k \le m$, such that the k-th coordinate of $A\mathbf{y}^T$ is the largest coordinate among all coordinates of $A\mathbf{y}^T$.

When a game matrix does not have a saddle point, both players do not have optimal pure strategies. However there always exists optimal mixed strategies for the players by the following minimax theorem due to von Neumann.

Theorem 2.1.10 (Minimax theorem)**.** *Let A be an $m \times n$ matrix. Then there exists real number $\nu \in \mathbb{R}$, mixed strategy for the row player $\mathbf{p} \in \mathbb{R}^m$, and mixed strategy for the column player $\mathbf{q} \in \mathbb{R}^n$ such that*

1. $\mathbf{p}A\mathbf{y}^T \geq \nu$, *for any* $\mathbf{y} \in \mathcal{P}^n$

2. $\mathbf{x}A\mathbf{q}^T \leq \nu$, *for any* $\mathbf{x} \in \mathcal{P}^m$

3. $\mathbf{p}A\mathbf{q}^T = \nu$

In the above theorem, the real number $\nu = \nu(A)$ is called the **value**, or the **security level**, of the game matrix A. The strategy $\mathbf{p}$ is called a **maximin strategy** for the row player and the strategy $\mathbf{q}$ is called a **minimax strategy** for the column player. The value ν of a matrix is unique. However maximin strategy and minimax strategy are in general not unique.

The maximin strategy $\mathbf{p}$ and the minimax strategy $\mathbf{q}$ are the optimal strategies for the row player and the column player respectively. It is because the row player may guarantee that his payoff is at least ν no matter how the column player plays by using the maximin strategy $\mathbf{p}$. This is also the reason why the value ν is called the security level. Similarly, the column player may guarantee that the payoff to the row player is at most ν, and thus his payoff is at least $-\nu$, no matter how the row player plays by using the minimax strategy $\mathbf{q}$. We will prove the minimax theorem in Section 3.4.

2.2 2×2 games

In this section, we study 2×2 game matrices closely and see how one can solve them, that means finding the maximin strategies for the row player, minimax strategies for the column player and the values of the game. First we look at a simple example.

Example 2.2.1 (Modified rock-paper-scissors)**.** *The rules of the modified rock-paper-scissors are the same as the ordinary rock-paper-scissors except that the row player can only show the gesture rock(R) or paper(P) but not scissors while the column player can only show the gesture scissors(S) or rock(R) but not paper. The game matrix of the game is*

$$\begin{array}{c} \\ R \\ P \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

It is easy to see that the game matrix does not have a saddle point. Thus there is no pure maximin or minimax strategy. To solve the game, suppose *the row player uses mixed strategy* $\mathbf{x} = (x, 1-x)$*. Consider*

$$\mathbf{x}A = (x, 1-x)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (x - (1-x), 1-x) = (2x-1, 1-x)$$

This shows that when the row player uses mixed strategy $\mathbf{x} = (x, 1-x)$*, then his payoff is* $2x-1$ *if the column player uses his 1st strategy scissors(S) and is* $1-x$ *if the column player uses his 2nd strategy rock(R). Now we solve the equation* $2x-1 = 1-x$ *and get* $x = \frac{2}{3}$*. One may see that if* $0 \leq x < \frac{2}{3}$*, then* $2x-1 < x-1$ *and a rational column player would use his 1st strategy scissors(S) and the payoff to the row player would be* $2x-1 < \frac{1}{3}$*. On the other hand, if* $\frac{2}{3} < x \leq 1$*, then* $2x-1 > 1-x$ *and a rational column player would use his 2nd strategy rock(R) and the payoff to the row player would be* $1-x < \frac{1}{3}$*. Now if* $x = \frac{2}{3}$*, that is if the row player uses the mixed strategy* $(\frac{2}{3}, \frac{1}{3})$*, then he may guarantee that his payoff is* $1-x = 2x-1 = \frac{1}{3}$ *no matter how the column player plays. This is the largest payoff he may guarantee and therefore the mixed strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *is the maximin strategy for the row player. Similarly, suppose the column player uses mixed strategy* $\mathbf{y} = (y, 1-y)$*. Consider*

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} y \\ -y+(1-y) \end{pmatrix}\begin{pmatrix} y \\ 1-2y \end{pmatrix}$$

If $0 \leq y < \frac{1}{3}$*, then* $y < 1-2y$ *and a rational row player would use his 2nd strategy paper(P) and his payoff would be* $1-2y > \frac{1}{3}$*. If* $\frac{1}{3} < y \leq 1$*, then* $y > 1-2y$ *and a rational row player would use his 1st strategy rock(R) and his payoff would be* $y > \frac{1}{3}$*. If* $y = \frac{1}{3}$*, then the payoff to the row player is always* $\frac{1}{3}$ *no matter how he plays. Therefore* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *is the minimax strategy for the column player and he may guarantee that the payoff to the row player is* $\frac{1}{3}$ *no matter how the row player plays. Moreover the value of the game is* $\nu = \frac{1}{3}$*. We summarize the above discussion in the following statements.*

1. *The row player may use his maximin strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *to guarantee that his payoff is* $\nu = \frac{1}{3}$ *no matter how the column player plays.*

2. *The column player may use his minimax strategy* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *to guarantee that the payoff to the row player is* $\nu = \frac{1}{3}$ *no matter how the row player plays.* □

Now we give the complete solutions to 2×2 games.

Theorem 2.2.2. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 *game matrix. Suppose* A *has no saddle point. Then*

1. *The value of the game is*

$$\nu = \frac{ad - bc}{a - b - c + d}$$

2. *The maximin strategy for the row player is*

$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$

3. *The minimax strategy for the column player is*

$$\mathbf{q} = \left(\frac{d - b}{a - b - c + d}, \frac{a - c}{a - b - c + d} \right)$$

Proof. Suppose the row player uses mixed strategy $\mathbf{x} = (x, 1 - x)$. Consider

$$\mathbf{x}A = (x, 1-x) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+c(1-x), bx+d(1-x)) = ((a-c)x+c, (b-d)x+d)$$

Now the payoff to the row player that he can guarantee is

$$\min\{(a - c)x + c, (b - d)x + d\}$$

Since A has no saddle point, we have $a - c$ and $b - d$ are of different sign and the maximum of the above minimum is obtained when

$$\begin{aligned} (a - c)x + c &= (b - d)x + d \\ \Rightarrow \qquad x &= \frac{d - c}{a - b - c + d} \end{aligned}$$

Note that x and $1 - x = \frac{a-b}{a-b-c+d}$ must be of the same sign because A has no saddle point. We must have $0 < x < 1$ and we conclude that the maximin strategy for the row player is

$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$

Similarly suppose the column player uses mixed strategy $\mathbf{y} = (y, 1-y)$. Consider

$$A\mathbf{y}^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} aq + b(1-y) \\ cq + d(1-y) \end{pmatrix} = \begin{pmatrix} (a-b)y + b \\ (c-d)y + d \end{pmatrix}$$

The column player may guarantee that the payoff to the row player is at most

$$\max\{(a-b)y + b, (c-d)y + d\}$$

The above quantity attains it minimum when

$$\begin{array}{rrcl} & (a-b)y + b & = & (c-d)y + d \\ \Rightarrow & y & = & \dfrac{d-b}{a-b-c+d} \end{array}$$

and the minimax strategy for the column player is

$$\mathbf{q} = \left(\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d} \right)$$

By calculating

$$\mathbf{p}A = \left(\frac{ad-bc}{a-b-c+d}, \frac{ad-bc}{a-b-c+d} \right) \text{ and } A\mathbf{q}^T = \begin{pmatrix} \frac{ad-bc}{a-b-c+d} \\ \frac{ad-bc}{a-b-c+d} \end{pmatrix}$$

we see that the maximum payoff that the row player may guarantee to himself and the minimum payoff to the row player that the column player may guarantee are both $\frac{ad-bc}{a-b-c+d}$. In fact the minimax theorem (Theorem 2.1.10) says that these two values must be equal. We conclude that the value of A is $\nu = \frac{ad-bc}{a-b-c+d}$. □

Note that the above formulas work only when A has no saddle point. If A has a saddle point, the vectors **p** and ıı obtained using the formulas may not be probability vectors.

Example 2.2.3. *Consider the modified rock-paper-scissors game (Example 2.2.1) with game matrix*

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The game matrix has no saddle point. By Theorem 2.2.2, the value of the game is

$$\nu = \frac{ad-bc}{a-b-c+d} = \frac{1\times 1 - 0\times(-1)}{1-0-(-1)+1} = \frac{1}{3}$$

the maximin strategy for the row player is

$$\begin{aligned}\mathbf{p} &= \left(\frac{d-c}{a-b-c+d}, \frac{a-b}{a-b-c+d}\right)\\ &= \left(\frac{1-(-1)}{1-0-(-1)+1}, \frac{1-0}{1-0-(-1)+1}\right)\\ &= \left(\frac{2}{3}, \frac{1}{3}\right)\end{aligned}$$

and the minimax strategy for the column player is

$$\begin{aligned}\mathbf{q} &= \left(\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d}\right)\\ &= \left(\frac{1-0}{1-0-(-1)+1}, \frac{1-(-1)}{1-0-(-1)+1}\right)\\ &= \left(\frac{1}{3}, \frac{2}{3}\right)\end{aligned}$$

□

Example 2.2.4. *In a game, each of the two players Andy and Bobby calls out a number simultaneously. Andy may call out either* 1 *or* -2 *while Bobby may call out either* 1 *or* -3*. Then Bobby pays* p *dollars to Andy where* p *is the product of the two numbers (Andy pays Bobby* $-p$ *dollars when* p *is negative). The game matrix of the game is*

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

The value of the game is

$$\nu = \frac{1\times 6-(-2)\times(-3)}{1-(-3)-(-2)+6} = 0$$

the maximin strategy for Andy is

$$\mathbf{p} = \left(\frac{6-(-2)}{1-(-3)-(-2)+6}, \frac{1-(-3)}{1-(-3)-(-2)+6}\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

and the minimax strategy for Bobby is

$$\mathbf{q} = \left(\frac{6-(-3)}{1-(-3)-(-2)+6}, \frac{1-(-2)}{1-(-3)-(-2)+6} \right) = \left(\frac{3}{4}, \frac{1}{4} \right)$$

□

We say that a two-person zero sum game is **fair** if its value is zero. The game in Example 2.2.4 is a fair game.

2.3 Games reducible to 2×2 games

To solve an $m \times n$ game matrix for $m, n > 2$ without saddle point, we may first remove the dominated rows or columns. A row dominates another if all its entries are larger than or equal to the corresponding entries of the other. Similarly, a column dominates another if all its entries are smaller than or equal to the corresponding entries of the other.

Definition 2.3.1. *Let $A = [a_{ij}]$ be an $m \times n$ game matrix.*

1. *We say that the k-th row is dominated by the r-th row if $a_{kj} \leq a_{rj}$ for any $j = 1, 2, \cdots, n$.*
2. *We say that the l-th column is dominated the s-th column if $a_{il} \geq a_{is}$ for any $i = 1, 2, \cdots, m$.*

We say that a row (column) is a **dominated row** *(***column***) if it is dominated by another row (column).*

If the k-th row of A is dominated by the r-th row, then for the row player, playing the r-th strategy is at least as good as playing the k-th strategy. Thus the k-th row can be ignored in finding the maximin strategy for the row player. Similarly the column player may ignore a dominated column when finding his minimax strategy.

Theorem 2.3.2. *Let A be an $m \times n$ game matrix. Suppose A has a dominated row or dominated column. Let A' be the matrix obtained by deleting a dominated row or dominated column from A. Then*

1. *The value of A' is equal to the value of A.*

2. *The players of A have maximin/minimax strategies which never use dominated row/column.*

More precisely, if the k-th row is a dominated row of A, A' is the $(m-1)\times n$ matrix obtained by deleting the k-th row from A, and $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ is a maximin strategy for the row player of A', then $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$ is a maximin strategy for the row player of A. Similarly, if the l-th column is a dominated row of A, A' is the $m \times (n-1)$ matrix obtained by deleting the l-th column from A, and $\mathbf{q}' = (q_1, \cdots, q_{l-1}, q_{l+1} \cdots, q_n) \in \mathcal{P}^{n-1}$ is a minimax strategy of A', then $\mathbf{q} = (q_1, \cdots, q_{l-1}, 0, q_{l+1}, \cdots, q_n) \in \mathcal{P}^n$ is a minimax strategy of A.

Proof. Suppose the k-th row of A is dominated by the r-th row and A' is obtained by deleting the k-th row from A. Let ν' be the value of A' and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy of A'. For any mixed strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$, define $\mathbf{x}' = (x'_1, \cdots, x'_{k-1}, x'_{k+1}, \cdots, x'_m) \in \mathcal{P}^{m-1}$ by

$$x'_i = \begin{cases} x_i & \text{if } i \neq r \\ x_k + x_r & \text{if } i = r \end{cases}$$

and we have

$$\mathbf{x}A\mathbf{q}^T \leq \mathbf{x}'A'\mathbf{q}^T \leq \nu'$$

Here the first inequality holds because the k-th is dominated by the r-th row and the second inequality holds because $\mathbf{q}$ is a minimax strategy of A'. Thus the value of A is less than or equal to ν'. On the other hand, let $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ be a maximin strategy of A' and let $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$. Then we have

$$\mathbf{p}A\mathbf{y}^T = \mathbf{p}'A'\mathbf{y}^T \geq \nu'$$

for any $\mathbf{y} \in \mathcal{P}^n$. It follows that the value of A is ν' and $\mathbf{p}$ is a maximin strategy of A. The proof of the second statement is similar. □

The removal of dominated rows or columns does not change the value of a game. The above theorem only says that there is at least one optimal strategy with zero probability at the dominated rows and columns. There may be other optimal strategies which have positive probability at the dominated rows or columns. However any optimal strategy must have zero probability at strictly dominated rows and columns. Here a row is strictly dominated by another row if all its entries are strictly smaller than the corresponding entries of the other. Similarly a column is strictly dominated by another column if all its entries are strictly larger than the corresponding entries of the other.

Example 2.3.3. *To solve the game matrix*

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -3 & 1 \\ -2 & 4 & 0 \end{pmatrix}$$

we may delete the second row since it is dominated by the first row and get the reduced matrix

$$A' = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 4 & 0 \end{pmatrix}$$

Then we may delete the third column since is dominated by the first column. Hence the matrix A is reduced to the 2×2 matrix

$$A'' = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$$

The value of this 2×2 matrix is 0.7. The maximin and minimax strategies are $(0.6, 0.4)$ and $(0.5, 0.5)$ respectively. Therefore the value of A is 0.7, a maximin strategy for the row player is $(0.6, 0, 0.4)$ and a minimax strategy for the column player is $(0.5, 0.5, 0)$. Note that we need to insert the zeros to the dominated rows and columns when writing down the maximin and minimax strategies for the players. □

2.4 $2 \times n$ and $m \times 2$ games

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

be a $2 \times n$ matrix. We are going to explain how to solve the game with game matrix A if there is no dominated row or column. Suppose the row player uses strategy $\mathbf{x} = (x, 1-x)$ for $0 \le x \le 1$. The payoff to the row player is given by

$$\begin{aligned} \mathbf{x}A &= (x, 1-x) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \\ &= (a_{11}x + a_{21}(1-x), a_{12}x + a_{22}(1-x), \cdots, a_{1n}x + a_{2n}(1-x)) \end{aligned}$$

Now we need to find the value of x so that the minimum

$$\min_{1\leq j\leq n}\{a_{1j}x + a_{2j}(1-x)\}$$

of the coordinates of $\mathbf{x}A$ attains its maximum. We may use graphical method to achieve this goal.

Step 1.
For each $1 \leq j \leq n$, draw the graph of

$$v = a_{1j}x + a_{2j}(1-x), \text{ for } 0 \leq x \leq 1$$

The graph shows the payoff to the row player if the column player uses the j-th strategy.

Step 2.
Draw the graph of

$$v = \min_{1\leq j\leq n}\{a_{1j}x + a_{2j}(1-x)\}$$

This is called the **lower envelope** of the graph.

Step 3.
Suppose (p, ν) is a maximum point of the lower envelope. Then ν is the value of the game and $\mathbf{p} = (p, 1-p)$ is a maximin strategy for the row player.

Step 4.
The solutions for $\mathbf{y} \in \mathcal{P}^n$ to the equation

$$A\mathbf{y}^T = \nu\mathbf{1}^T$$

where $\mathbf{1} = (1, 1)$, give the minimax strategy for the column player.

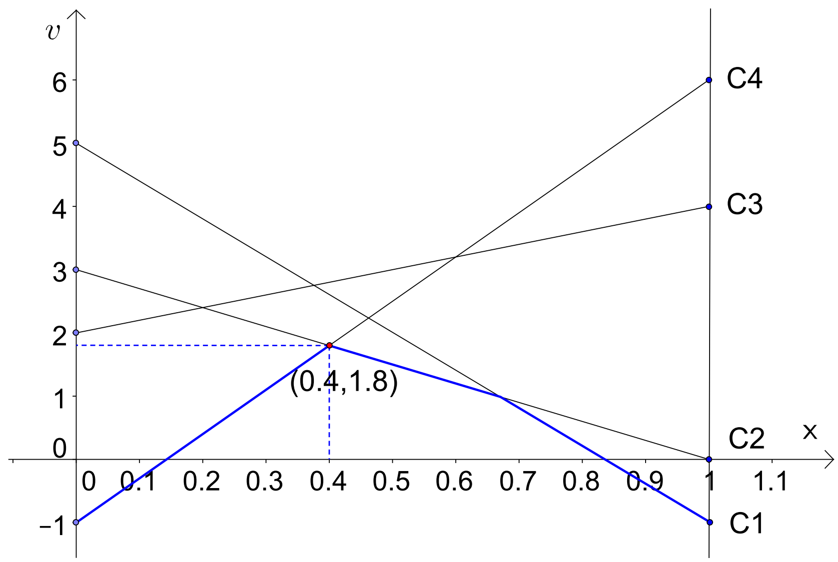
Example 2.4.1. *Solve the* 2×4 *game matrix*

$$A = \begin{pmatrix} -1 & 0 & 4 & 6 \\ 5 & 3 & 2 & -1 \end{pmatrix}$$

Solution.

Step 1. Draw the graph of

$$\begin{cases} C1: & v = -x + 5(1-x) \\ C2: & v = 3(1-x) \\ C3: & v = 4x + 2(1-x) \\ C4: & v = 6x - (1-x) \end{cases}$$



Step 2. Draw the lower envelope (blue polygonal curve).

Step 3. The maximum point of the lower envelope is the intersection point of $C2$ and $C4$. By solving

$$\begin{cases} C2: & v = 3(1-x) \\ C4: & v = 6x - (1-x) \end{cases}$$

we obtain the maximum point $(p, \nu) = (0.4, 1.8)$ of the lower envelope.

Step 4. Find the minimax strategies for the column player by solving

$$\begin{pmatrix} 0 & 6 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.8 \end{pmatrix}$$

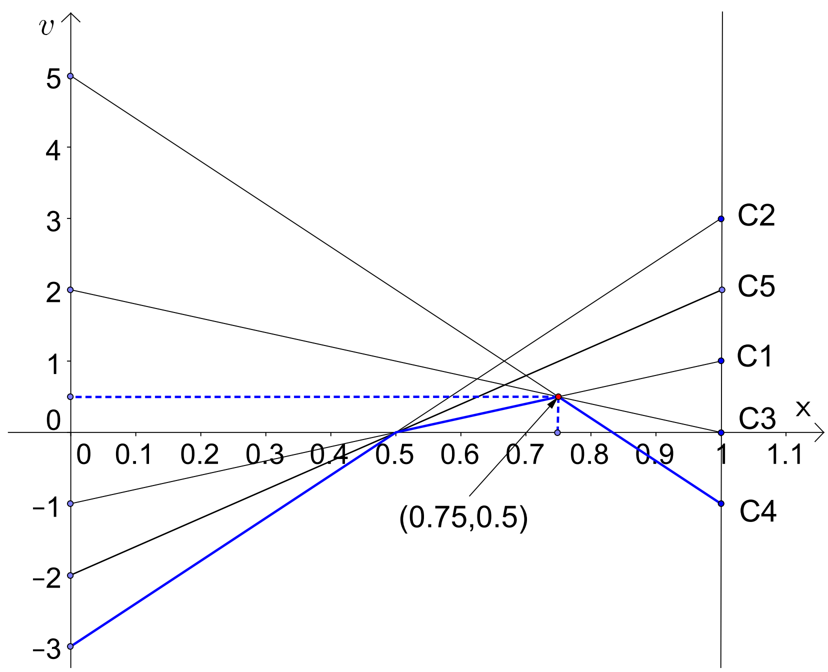
and get $y_2 = 0.7$ and $y_4 = 0.3$.

Therefore the value of the game is $\nu = 1.8$. The maximin strategy for the row player is $\mathbf{p} = (0.4, 0.6)$ and the minimax strategy for the column player is $\mathbf{q} = (0, 0.7, 0, 0.3)$. $\square$

Example 2.4.2. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ -1 & -3 & 2 & 5 & -2 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



By solving

$$\begin{cases} C1: & v = x - (1-x) \\ C3: & v = 2(1-x) \\ C4: & v = -x + 5(1-x) \end{cases}$$

we see that the maximum point of the lower envelope is $(p, \nu) = (0.75, 0.5)$. Thus the maximin strategy for the row player is $(0.75, 0.25)$ and the value of the game is $\nu = 0.5$. To find minimax strategies for the column player, we solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix}$$

Note that we have added the equation $y_1+y_3+y_4 = 1$ to exclude the solutions which are not probability vectors. (Explain why we didn't do it in Example 2.4.1.) Using row operation, we obtain the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0.5 \\ -1 & 2 & 5 & 0.5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0.5 \\ 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The non-negative solution to the system of equations is

$$(y_1, y_3, y_4) = (0.5 + t, 0.5 - 2t, t) \text{ for } 0 \leq t \leq 0.25$$

Therefore the column player has minimax strategies

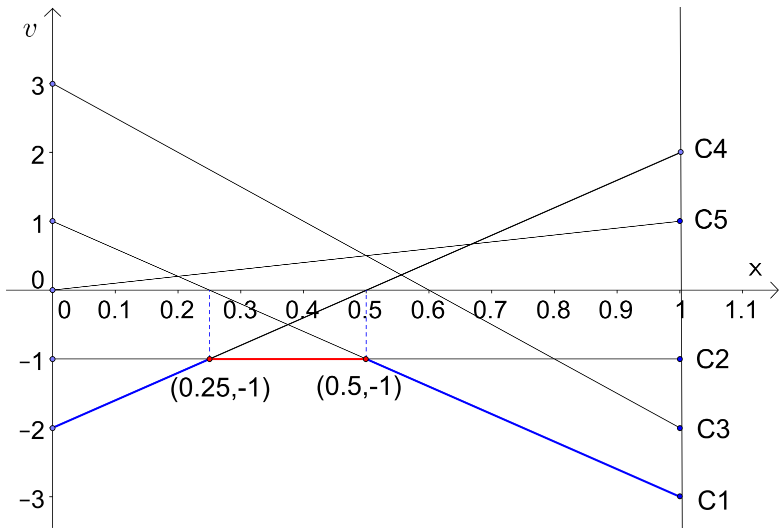
$$\mathbf{q} = (0.5 + t, 0, 0.5 - 2t, t, 0) \text{ for } 0 \leq t \leq 0.25$$

In particular, $(0.5, 0, 0.5, 0, 0)$ and $(0.75, 0, 0, 0.25, 0)$ are minimax strategies for the column player. □

Example 2.4.3. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} -3 & -1 & -2 & 2 & 1 \\ 1 & -1 & 3 & -2 & 0 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



The maximum points of the lower envelope are points lying on the line segment joining $(0.25, -1)$ and $(0.5, -1)$. Thus the value of the game is $\nu = -1$. The maximin strategies for the row player are

$$\mathbf{p} = (p, 1-p) \text{ for } 0.25 \leq p \leq 0.5$$

and the minimax strategy for the column player is

$$\mathbf{q} = (0, 1, 0, 0, 0)$$

□

Next we consider $m \times 2$ games. There are two methods to solve such games.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy for the column player. Draw the upper envelope

$$v = \max_{1 \leq i \leq m} \{a_{i1}y + a_{i2}(1-y)\}$$

Suppose the minimum point of the upper envelope is (q, ν). Then the value of the game is ν and the minimax strategy for the column player is $\mathbf{q} = (q, 1-q)$. Moreover the maximum strategies for the row player are the solutions for $\mathbf{x} \in \mathcal{P}^m$ to the equation

$$\mathbf{x}A = \nu\mathbf{1} = (\nu, \nu)$$

Method 2.
Solve the game with $2 \times m$ game matrix $-A^T$. Then

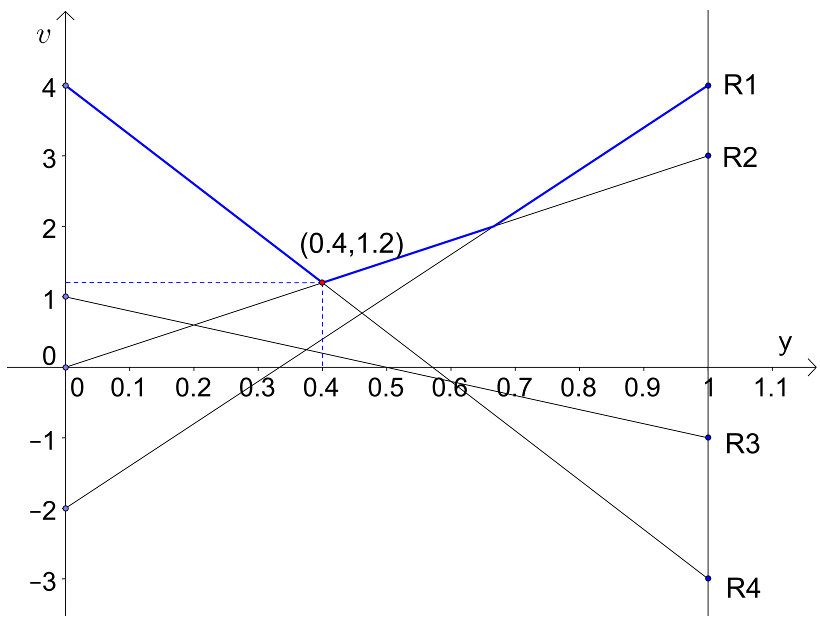
value of $A = -$ value of $-A^T$
maximin strategy of $A =$ minimax strategy of $-A^T$
minimax strategy of $A =$ maximin strategy of $-A^T$

Example 2.4.4. *Solve the* 4×2 *game matrix*

$$A = \begin{pmatrix} 4 & -2 \\ 3 & 0 \\ -1 & 1 \\ -3 & 4 \end{pmatrix}$$

Solution.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy of the column player. The upper envelope is



Solving

$$\begin{cases} R2: & v = 3(1-y) \\ R4: & v = -3y + 4(1-y) \end{cases}$$

the minimum point of the upper envelope is $(q, \nu) = (0.4, 1.2)$. Now the row player would only use the 2nd and 4th strategy and we solve

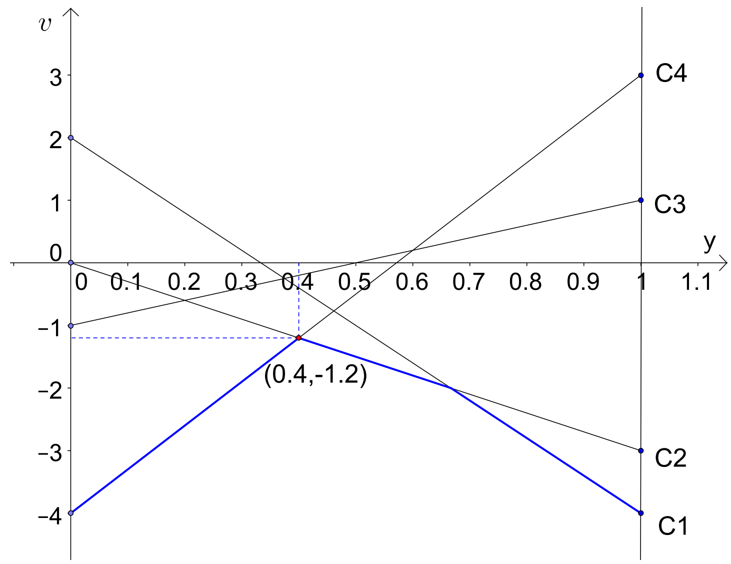
$$(x_2, x_4) \begin{pmatrix} 3 & 0 \\ -3 & 4 \end{pmatrix} = (1.2, 1.2)$$

which gives $(x_2, x_4) = (0, 7, 0.3)$. Therefore the value of the game is $\nu = 1.2$, the maximin strategy for the row player is $\mathbf{p} = (0, 0.7, 0, 0.3)$ and the minimax strategy for the column player is $\mathbf{q} = (0.4, 0.6)$.

Method 2.
Consider

$$-A^T = \begin{pmatrix} -4 & -3 & 1 & 3 \\ 2 & 0 & -1 & -4 \end{pmatrix}$$

Draw the lower envelope



We see that the value of $-A^T$ is -1.2 and the maximin strategy of $-A^T$ is $(0.4, 0.6)$. Solving

$$\begin{pmatrix} -3 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1.2 \\ -1.2 \end{pmatrix}$$

We get $x_2 = 0.7$ and $x_4 = 0.3$. Thus the minimax strategy of $-A^T$ is $(0, 0.7, 0, 0.3)$. Therefore

$$\begin{aligned} &\text{value of } A = -\text{ value of } -A^T = 1.2 \\ &\text{maximin strategy of } A = \text{minimax strategy of } -A^T = (0, 0.7, 0, 0.3) \\ &\text{minimax strategy of } A = \text{maximin strategy of } -A^T = (0.4, 0.6) \end{aligned}$$

□

Theorem 2.4.5 (Principle of indifference)**.** *Let A be an $m \times n$ game matrix. Suppose ν is the value of A, $\mathbf{p} = (p_1, \cdots, p_m)$ be a maximin strategy for the row player and $\mathbf{q} = (q_1, \cdots, q_n)$ be a minimax strategy for the column player. For any $k = 1, 2, \cdots, m$, if $p_k > 0$, then $\sum_{j=1}^{n} a_{kj} q_j = \nu$. In particular, when the column player uses his minimax strategy $\mathbf{q}$, then the payoff to the row* *player are indifferent among all his k-th strategies with $p_k > 0$. Similarly, for any $l = 1, 2, \cdots, n$, if $q_l > 0$, then $\sum_{i=1}^{m} a_{il} p_i = \nu$. In particular, when the row player uses his maximin strategy $\mathbf{p}$, then the payoff to the row player are indifferent among all the l-th strategies of the column player with $q_l > 0$.*

Proof. For each $k = 1, 2, \cdots, m$, we have

$$\sum_{j=1}^{n} a_{kj} q_j \leq \nu$$

since $\mathbf{q}$ is a minimax strategy for the column player. On the other hand,

$$\nu = \mathbf{p} A \mathbf{q}^T = \sum_{k=1}^{m} p_k \left(\sum_{j=1}^{n} a_{kj} q_j \right) \leq \sum_{k=1}^{m} p_k \nu = \nu$$

Thus we have

$$p_k \sum_{j=1}^{n} a_{kj} q_j = p_k \nu$$

for any $k = 1, 2, \cdots, m$. Therefore

$$\sum_{j=1}^{n} a_{kj} q_j = \nu$$

whenever $p_k > 0$. The proof of the second statement is similar. □

Exercise 2

1. Find the values of the following game matrices by finding their saddle points

(a) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 5 & -3 & -3 \\ 0 & 1 & 3 & -1 \\ -3 & -1 & 2 & -5 \\ 2 & -4 & 0 & -2 \end{pmatrix}$

2. Solve the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column.

(a) $\begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & -3 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 2 & -1 \\ 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -2 & 3 \\ 3 & -1 & 4 \\ 0 & 3 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$

3. Raymond holds a black 2 and a red 9. Calvin holds a red 3 and a black 8. Each of them chooses one of the cards from his hand and then two players show the chosen cards simultaneously. If the chosen cards are of the same colour, Raymond wins and Calvin wins if the cards are of different colours. The loser pays the winner an amount equal to the number on the winner's card. Write down the game matrix, find the value of the game and the optimal strategies of the players.

4. Alex and Becky point fingers to each other, with either one finger or two fingers. If they match with one finger, Becky pays Alex 3 dollars. If they match with two fingers, Becky pays Alex 11 dollars. If they don't match, Alex pays Becky 1 dollar.

   (a) Find the optimal strategies for Alex and Becky.

   (b) Suppose Alex pays Becky k dollars as a compensation before the game. Find the value of k to make the game fair.

5. Player I and II choose integers i and j respectively where $1 \leq i, j \leq 7$. Player II pays Player I one dollar if $|i - j| = 1$. Otherwise there is no payoff. Write down the game matrix of the game, find the value of the game and the optimal strategies for the players.

6. Use the principle of indifference to solve the game with game matrix

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. In the Mendelsohn game, two players choose an integer from 1 to 5 simultaneously. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1 dollar from its opponent. The player that chooses a number two or more larger than his opponent loses 2 dollars to its opponent. Find the game matrix and solve the game.

8. Let
$$A = \begin{pmatrix} -3 & 1 \\ c & -2 \end{pmatrix}$$
where c is a real number.
   (a) Find the range of values of c such that A has a saddle point.
   (b) Suppose the zero sum game with game matrix A is a fair game.
      (i) Find the value of c.
      (ii) Find the maximin strategy for the row player and the minimax strategy for the column player.

9. Prove that if A is a skewed symmetric matrix, that is, $A^T = -A$, then the value of A is zero.

10. Let n be a positive integer and
$$D = \begin{pmatrix} \lambda_1 & & & \text{\huge 0} \\ & \lambda_2 & & \\ & & \ddots & \\ \text{\huge 0} & & & \lambda_n \end{pmatrix}$$
be an $n \times n$ diagonal matrix where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
   (a) Suppose $\lambda_1 \leq 0$ and $\lambda_n > 0$. Find the value of the zero sum game with game matrix D.
   (b) Suppose $\lambda_1 > 0$. Solve the zero sum game with game matrix D.

3 Linear programming and maximin theorem

3.1 Linear programming

In this chapter we study two-person zero sum game with $m \times n$ game matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Suppose the row player uses strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$. Then the column player would use his j-th strategy such that

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m$$

is minimum among $j = 1, 2, \cdots, n$. Thus the payoff to the row player that he can guarantee is

$$\min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

Hence if the above expression attains its maximum at $\mathbf{x} = \mathbf{p} \in \mathcal{P}^m$, then $\mathbf{p}$ is a maximin strategy for the row player. Moreover, the value of the game is

$$\nu = \max_{\mathbf{x} \in \mathcal{P}^m} \min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

By introducing a new variable v, we can restate the **maximin problem**, that is finding a maximin strategy, as the following linear programming problem

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

Similarly, to find a minimax strategy for the column player, we need to solve the following **minimax** problem

$$
\begin{array}{rl}
\min & v \\
\text{subject to} & a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq v \\
& a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq v \\
& \vdots \\
& a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq v \\
& q_1 + q_2 + \cdots + q_n = 1 \\
& q_1, q_2, \cdots, q_n \geq 0
\end{array}
$$

To solve the maximin and minimax problems, first we transform them to a pair of primal and dual problems.

Definition 3.1.1 (Primal and dual problems). *A linear programming problem in the following form is called a* **primal problem**.

$$
\begin{array}{rl}
\max & f(y_1, \cdots, y_n) = \sum_{j=1}^{n} c_j y_j + d \\
\textit{subject to} & \sum_{j=1}^{n} a_{ij} y_j \leq b_i, \ i = 1, 2, \cdots, m \\
& y_1, y_2, \cdots, y_n \geq 0
\end{array}
$$

The **dual problem** *associated to the above primal problem is*

$$
\begin{array}{rl}
\min & g(x_1, \cdots, x_m) = \sum_{i=1}^{m} b_i x_i + d \\
\textit{subject to} & \sum_{i=1}^{m} a_{ij} x_i \geq c_j, \ j = 1, 2, \cdots, n \\
& x_1, x_2, \cdots, x_m \geq 0
\end{array}
$$

Here $x_1, \cdots, x_m, y_1, \cdots, y_n$ *are variables, and* $a_{ij}, b_i, c_j, d,\ i = 1, 2, \cdots, m,$ $j = 1, 2, \cdots, n,$ *are constants. The linear functions* f *and* g *are called* **objective functions**. *The primal problem and the dual problem can be written in the following matrix forms*

Primal problem	max *subject to* 	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$ $\mathbf{y} \geq \mathbf{0}$
Dual problem	min *subject to* 	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$ $\mathbf{x} \geq \mathbf{0}$

Here $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ *are variable vectors,* A *is an* $m \times n$ *constant matrix,* $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ *are constant vectors and* $d \in \mathbb{R}$ *is a real constant. The inequality* $\mathbf{u} \leq \mathbf{v}$ *for vectors* $\mathbf{u}, \mathbf{v}$ *means each of the coordinates of* $\mathbf{v} - \mathbf{u}$ *is non-negative.*

For primal and dual problems, we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$. In other words, all variables are non-negative. From now on, we will not write down the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$ for primal and dual problems and it is understood that all variables are non-negative.

Definition 3.1.2. *Suppose we have a pair of primal and dual problems.*

1. *We say that a vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **feasible** *if it satisfies the constraints of the problem. We say that the primal problem (or the dual problem) is feasible there exists a feasible vector for the problem.*

2. *We say that the primal problem, (or the dual problem), is* **bounded** *if the objective function is bounded above, (or below) on the set of feasible vectors.*

3. *We say that a feasible vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **optimal** *if the objective function* f *(or* g*) attains its maximum (or minimax) at* $\mathbf{x}$ *(or* $\mathbf{y}$*) on the set of feasible vectors.*

Theorem 3.1.3. *Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible vectors in the dual and primal problems respectively. Then*

$$f(\mathbf{y}) \leq g(\mathbf{x})$$

Proof. We have

$$\begin{aligned} f(\mathbf{y}) &= \mathbf{c}\mathbf{y}^T + d \\ &\leq \mathbf{x}A\mathbf{y}^T + d \quad \text{(since } \mathbf{x} \text{ is feasible and } \mathbf{y} \geq \mathbf{0}) \\ &\leq \mathbf{x}\mathbf{b}^T + d \quad \text{(since } \mathbf{y} \text{ is feasible and } \mathbf{x} \geq \mathbf{0}) \\ &= g(\mathbf{x}) \end{aligned}$$

□

The theorem above has a simple and important consequence that the primal problem is bounded if the dual problem associated with it has a feasible vector, and vice verse.

Theorem 3.1.4. *Suppose we have a pair of primal and dual problems.*

1. *If the primal problem is feasible, then the dual problem is bounded.*

2. *If the dual problem is feasible, then the primal problem is bounded.*

3. *If both problems are feasible, then both problems are solvable, that is, there exists optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problems respectively. Moreover we have* $f(\mathbf{p}) \leq g(\mathbf{q})$.

Proof. For the first statement, suppose the primal problem has a feasible vector $\mathbf{q}$. Then for any feasible vector $\mathbf{x}$ of the dual problem, we have $g(\mathbf{x}) \geq f(\mathbf{q})$ by Theorem 3.1.3. Hence the dual problem is bounded. The proof of the second statement is similar. For the third statement, suppose both problems are feasible. Then both problems are bounded by the first two statements. Observe that the set of feasible vectors is closed. It follows that the optimal values of the objective functions f and g are attainable. Therefore there exists optimal vectors $\mathbf{p}$ and $\mathbf{q}$ for the dual and primal problems respectively and $f(\mathbf{q}) \leq g(\mathbf{p})$ by Theorem 3.1.3. □

Furthermore we have the following important theorem in linear programming concerning the solutions to the primal and dual problems.

Theorem 3.1.5. *Suppose both the dual problem and the primal problem are feasible. Then there exist optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problem respectively, and we have*

$$f(\mathbf{q}) = g(\mathbf{p})$$

Proof. We have proved the solvability of the problems. The equality $f(\mathbf{q}) = g(\mathbf{p})$ can be proved using minimax theorem and we omit the proof here. □

3.2 Transforming maximin problem to dual problem

To find a maximin strategy for the row player of a two-person zero sum game, we have seen in the previous section that we need to solve the following maximin problem.

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

which can be written into following matrix form

$$\begin{array}{rl} \max & v \\ \text{subject to} & \mathbf{p}A \geq v\mathbf{1} \\ & \mathbf{p}\mathbf{1}^T = 1 \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$. We solve the above maximin problem in the following two steps.

1. Transform the maximin problem to a dual problem.

2. Use simplex method to solve the dual problem.

In this section, we are going to discuss how to transform a maximin problem to a dual problem. Note that the maximin problem is neither a primal nor dual problem because there is a constraint $p_1+p_2+\cdots+p_m = 1$ which is not allowed and we do not have the constraint $v \geq 0$. To transform the maximin problem into a dual problem, first we add a constant k to each entry of A so that the value of the game matrix is positive. Secondly, we let

$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

Then to maximize v is the same as minimizing

$$x_1 + x_2 + \cdots + x_m = \frac{p_1 + p_2 + \cdots + p_m}{v} = \frac{1}{v}$$

Moreover for each $j = 1, 2, \cdots, n$, the constraint

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m \geq v$$

is equivalent to

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq 1$$

and the maximin problem would become a dual problem. We summarize the above procedures as follows.

1. First, add a constant k to each entry of A so that every entry of A is positive. (This is done to make sure that the value of the game matrix is positive.)

2. Let
$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

3. Write down the dual problem
$$\begin{array}{rl} \min & g(x_1, x_2, \cdots, x_m) = x_1 + x_2 + \cdots + x_m \\ \text{subject to} & a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq 1 \\ & a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq 1 \\ & \vdots \\ & a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq 1 \end{array}$$
(Note that we always have the constraints $x_1, x_2, \cdots, x_m \geq 0$) or in matrix form
$$\begin{array}{rl} \min & g(\mathbf{x}) = \mathbf{x}\mathbf{1}^T \\ \text{subject to} & \mathbf{x}A \geq \mathbf{1} \end{array}$$
where $\mathbf{1} = (1, 1 \cdots, 1) \in \mathbb{R}^m$.

4. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ is an optimal vector of the dual problem and
$$d = g(\mathbf{x}) = x_1 + x_2 + \cdots + x_m$$
is the minimum value. Then
$$\mathbf{p} = \frac{\mathbf{x}}{d} = \left(\frac{x_1}{d}, \frac{x_2}{d}, \cdots, \frac{x_m}{d}\right)$$
is a maximin strategy for the row player and the value of the game matrix A is
$$v = \frac{1}{d} - k$$

To find the minimax strategy for the column player, we need to solve the following minimax problem.

$$\begin{array}{rl} \min & v \\ \text{subject to} & A\mathbf{q}^T \leq v\mathbf{1}^T \\ & \mathbf{1}\mathbf{q}^T = 1 \\ & \mathbf{q} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$. If we assume that $v > 0$, the above optimization problem can be transformed to the following primal problem by taking $y_j = \frac{q_j}{v}$ for $j = 1, 2, \cdots, n$.

$$\begin{array}{rl} \max & f(\mathbf{y}) = \mathbf{1}\mathbf{y}^T \\ \text{subject to} & A\mathbf{y} \leq \mathbf{1}^T \end{array}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. (Note that we always have the constraint $\mathbf{y} \geq \mathbf{0}$ for primal problem.) Suppose $\mathbf{y}$ is an optimal vector for the above primal problem. Then $\mathbf{q} = \frac{\mathbf{y}}{d}$ is a minimax strategy for the column player.

3.3 Simplex method

We have seen that a pair of maximin and minimax problems can be transformed to a pair of dual and primal problems. In this section, we will show how to use simplex method to solve the dual and primal problems simultaneously. Recall that the primal and dual problems are optimization problems of the following forms. Please be reminded that we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

Primal problem	max subject to	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$
Dual problem	min subject to	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$

We describe the **simplex method** as follows.

Step 1. Introduce new variables $x_{m+1}, \cdots, x_{m+m}, y_{n+1}, \cdots, y_{n+m}$ which are called **slack variables** and set up the tableau

	y_1	$\cdots$	y_n	-1	
x_1	a_{11}	$\cdots$	a_{1n}	b_1	$= -y_{n+1}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	b_m	$= -y_{n+m}$
-1	c_1	$\cdots$	c_n	$-d$	$= f$
	$\parallel$	$\cdots$	$\parallel$	$\parallel$	
	x_{m+1}	$\cdots$	x_{m+n}	g	

Step 2.

(i) If $c_1, c_2, \cdots, c_n \leq 0$, then the solution to the problems are

Primal problem	maximum value of $f = d$ $y_1 = y_2 = \cdots = y_n = 0,$ $y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m$
Dual problem	minimum value of $g = d$ $x_1 = x_2 = \cdots = x_m = 0,$ $x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_m$

(ii) Otherwise go to step 3.

Step 3. Choose $l = 1, 2, \cdots, n$ such that $c_l > 0$.

(i) If $a_{il} \leq 0$ for all $i = 1, 2, \cdots, m$, then the problems are unbounded (because y_l can be arbitrarily large) and there is no solution.

(ii) Otherwise choose $k = 1, 2, \cdots, m$, such that

$$\frac{b_k}{a_{kl}} = \min_{a_{il}>0} \left\{ \frac{b_i}{a_{il}} \right\}$$

Step 4. Pivot on the entry a_{kl} and swap the variables at the pivot row with the variables at the pivot column. The **pivoting operation** is performed as follows.

	y_l	y_j	
x_k	a^*	b	$= -y_{n+k}$
x_i	c	d	$= -y_{n+i}$
	$\parallel$	$\parallel$	
	x_{m+l}	x_{m+j}	

$\longrightarrow$

	y_{n+k}	y_j	
x_{m+l}	$\frac{1}{a}$	$\frac{b}{a}$	$= -y_l$
x_i	$-\frac{c}{a}$	$d - \frac{bc}{a}$	$= -y_{n+i}$
	$\parallel$	$\parallel$	
	x_k	x_{m+j}	

Step 5. Go to Step 2.

To understand how the simplex method works, we introduce basic forms of linear programming problem.

Definition 3.3.1 (Basic form)**.** *A* **basic form** *of a pair of primal and dual problems is a problem of the form*

Primal basic form	max *subject to*	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ $\mathbf{y} \geq \mathbf{0}$
Dual basic form	min *subject to*	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ $\mathbf{x} \geq \mathbf{0}$

where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ *and* $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$*. The pair of basic forms can be represented by the tableau*

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

The variables at the rightmost column and at the bottom row are called **basic variables***. The other variables at the leftmost columns and at the top row are called* **independent/non-basic variables***.*

A pair of primal and dual problems may be expressed in basic form in many different ways. The pivot operation changes one basic form of the pair of primal and dual problems to another basic form of the same pair of problems, and swaps one basic variable with one independent variable.

Theorem 3.3.2. *The basic forms before and after a pivot operation are equivalent.*

Proof. The tableau before the pivot operation

	y_l	y_j	
x_k	a^*	b	$= -y_{n+k}$
x_i	c	d	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_{m+l}	x_{m+j}	

is equivalent to the system of equations

$$\begin{cases} ax_k + cx_i = x_{m+l} \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} ay_l + by_j = -y_{n+k} \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} -x_{m+l} + cx_i = -ax_k \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} y_{n+k} + by_j = -ay_l \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ b\left(\frac{1}{a}x_{m+l} - \frac{c}{a}x_i\right) + dx_i = x_{m+j} \end{cases}$$

$$\text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ c\left(\frac{1}{a}y_{n+k} + \frac{b}{a}y_j\right) + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ \frac{b}{a}x_{m+l} + \left(d - \frac{bc}{a}\right)x_i = x_{m+j} \end{cases}$$

$$\text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ -\frac{c}{a}y_{n+k} + \left(d - \frac{bc}{a}\right)y_j = -y_{n+i} \end{cases}$$

which is equivalent to the tableau

	y_{n+k}	y_j	
x_{m+l}	$\frac{1}{a}$	$\frac{b}{a}$	$= -y_l$
x_i	$-\frac{c}{a}$	$d - \frac{bc}{a}$	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_k	x_{m+j}	

The above calculation shows that the constraints before and after a pivot operation are equivalent, and the values of the objective functions f and g for any given $x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}$ and $y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}$ satisfying the constraints remain unchanged. $\square$

For each pair of basic forms, there associates a pair of basic solutions which will be defined below. Note that the basic solutions are not really solutions to the primal and dual problems because basic solutions are not necessarily feasible.

Definition 3.3.3 (Basic solution)**.** *Suppose we have a pair of basic forms represented by the tableau*

$$\begin{array}{c|ccc|c|l} & y_1 & \cdots & y_n & -1 & \\ \hline x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\ x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\ \hline -1 & c_1 & \cdots & c_n & -d & = f \\ & \shortparallel & \cdots & \shortparallel & \shortparallel & \\ & x_{m+1} & \cdots & x_{m+n} & g & \end{array}$$

The **basic solution** *to the basic form is*

$$\begin{aligned} &x_1 = x_2 = \cdots = x_m = 0, x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_n \\ &y_1 = y_2 = \cdots = y_n = 0, y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m \end{aligned}$$

The basic solutions are obtained by setting the independent variables, that is the variables at the top and at the left, to be 0 and then solving for the basic variables, that is the variables at the bottom and at the right, by the constraints.

The basic solutions always satisfy the equalities in the constraints, but they may not be feasible since some variables may have negative values. However if both the dual and primal basic solutions are feasible, then they must be optimal.

Theorem 3.3.4. *Suppose we have a pair of basic forms.*

1. *The basic solution to the primal basic form is feasible if and only if* $b_1, b_2, \cdots, b_m \geq 0$*.*

2. *The basic solution to the dual basic form is feasible if and only if* $c_1, c_2, \cdots, c_n \le 0$.

3. *The pair of basic solutions are optimal if* $b_1, \cdots, b_m \ge 0$ *and* $c_1, \cdots, c_n \le 0$.

Proof. Observe that the basic solutions always satisfy the equalities $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ and $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ of the constraints.

1. The basic solution to the primal basic form is $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$. Thus it is feasible if and only if all the variables are non-negative which is equivalent to $b_1, b_2, \cdots, b_m \ge 0$.

2. The basic solution to the dual basic form is $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$. Thus it is is feasible if and only if all the variables are non-negative which is equivalent to $c_1, c_2, \cdots, c_n \le 0$.

3. Suppose $b_1, b_2, \cdots, b_m \ge 0$ and $c_1, c_2, \cdots, c_n \le 0$. For any feasible vectors $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n})$ of the dual basic form and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m})$ of the primal basic form, we have

$$\begin{aligned} f(y_1, \cdots, y_n) &= (c_1, \cdots, c_n)(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)A(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)(b_1, \cdots, b_m)^T + d \\ &= g(x_1, \cdots, x_m) \end{aligned}$$

   On the other hand, the basic solutions $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$ and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$ are feasible and

$$f(0, \cdots, 0) = d = g(0, \cdots, 0)$$

   Therefore f attains its maximin and g attains its minimum at the basic solutions.

□

In practice, we do not write down the basic variables. We would swap the variables at the left and at the top when preforming pivot operation. One may find the basic and independent variables by referring to the following table.

	Left	Top
x_i	x_i is independent variable y_{n+i} is basic variable	x_i is basic variable y_{n+i} is independent variable
y_j	y_j is basic variable x_{m+j} is independent variable	y_j is independent variable x_{m+j} is basic variable

In other words, when we write down a tableau of the form

$$\begin{array}{c|cc|c} & x_i & y_l & -1 \\ \hline y_j & & & b_i \\ x_k & \multicolumn{2}{c|}{A} & b_k \\ \hline -1 & c_j & c_l & -d \end{array}$$

the basic solution associated with it is

$$\begin{aligned} &x_i = -c_j, x_k = 0, x_{m+j} = 0, x_{m+l} = -c_l \\ &y_j = b_i, y_l = 0, y_{n+i} = 0, y_{n+k} = b_k \end{aligned}$$

and the genuine tableau is

$$\begin{array}{c|cc|c} & y_{n+i} & y_l & -1 \\ \hline x_{m+j} & & & b_i = -y_j \\ x_k & \multicolumn{2}{c|}{A} & b_k = -y_{n+k} \\ \hline -1 & c_j & c_l & -d \\ & \shortparallel & \shortparallel & \\ & x_i & x_{m+l} & \end{array}$$

Example 3.3.5. *Solve the following primal problem.*

$$\begin{array}{rl} \max & f = 6y_1 + 4y_2 + 5y_3 + 150 \\ \textit{subject to} & 2y_1 + y_2 + y_3 \leq 180 \\ & y_1 + 2y_2 + 3y_3 \leq 300 \\ & 2y_1 + 2y_2 + y_3 \leq 240 \end{array}$$

Solution. Set up the tableau and perform pivot operations successively. The pivoting entries are marked with asterisks.

	y_1	y_2	y_3	-1
x_1	2^*	1	1	180
x_2	1	2	3	300
x_3	2	2	1	240
-1	6	4	5	-150

$\longrightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	90
x_2	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	210
x_3	-1	1^*	0	60
-1	-3	1	2	-690

$\longrightarrow$

	x_1	x_3	y_3	-1
y_1	1	$-\frac{1}{2}$	$\frac{1}{2}$	60
x_2	1	$-\frac{3}{2}$	$\frac{5}{2}^*$	120
y_2	-1	1	0	60
-1	-2	-1	2	-750

$\longrightarrow$

	x_1	x_3	x_2	-1
y_1	$\frac{4}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	36
y_3	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{2}{5}$	48
y_2	-1	1^*	0	60
-1	$-\frac{14}{5}$	$\frac{1}{5}$	$-\frac{4}{5}$	-846

$\longrightarrow$

	x_1	y_2	x_2	-1
y_1	$\frac{3}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	48
y_3	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	84
x_3	-1	1	0	60
-1	$-\frac{13}{5}$	$-\frac{1}{5}$	$-\frac{4}{5}$	-858

The independent variables are y_2, y_4, y_5 and the basic variables are y_1, y_3, y_6. The basic solution is

$$y_2 = y_4 = y_5 = 0, y_1 = 48, y_3 = 84, y_6 = 60$$

Thus an optimal vector for the primal problem is

$$(y_1, y_2, y_3) = (48, 0, 84)$$

The maximum value of f is 858.

We may also write down an optimal solution to the dual problem. The dual problem is

$$\begin{array}{rl} \min & g = 180x_1 + 300x_2 + 240x_3 + 150 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 \geq 6 \\ & x_1 + 2x_2 + 2x_3 \geq 4 \\ & x_1 + 3x_2 + x_3 \geq 5 \end{array}$$

From the last tableau, the independent variables are x_3, x_4, x_6 and the basic variables are x_1, x_2, x_5. The basic solution is

$$x_3 = x_4 = x_6 = 0, x_1 = \frac{13}{5}, x_2 = \frac{4}{5}, x_5 = \frac{1}{5}$$

Therefore an optimal vector for the dual problem is

$$(x_1, x_2, x_3) = \left(\frac{13}{5}, \frac{4}{5}, 0\right)$$

The minimum value of g is 858 which is equal to the maximum value of f.□

To use simplex method solving a game matrix, first we add a constant k to every entry so that the entries are all non-negative and there is no zero column. This is done to make sure that the value of the new matrix is positive. Then we take $\mathbf{b} = (1, \cdots, 1) \in \mathbb{R}^m$, $\mathbf{c} = (1, \cdots, 1) \in \mathbb{R}^n$ to set up the initial tableau

$$\begin{array}{c|ccc|c} & y_1 & \cdots & y_n & \\ \hline x_1 & a_{11} & \cdots & a_{1n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m & a_{m1} & \cdots & a_{mn} & 1 \\ \hline & 1 & \cdots & 1 & 0 \end{array}$$

and apply the simplex algorithm. Then the value of the game matrix is

$$\nu = \frac{1}{d} - k$$

where d is the maximum value of f or the minimum value of g, and k is the constant which is added to the game matrix at the beginning. A maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}\mathbf{x} = \frac{1}{d}(x_1, x_2, \cdots, x_m)$$

and a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}\mathbf{y} = \frac{1}{d}(y_1, x_2, \cdots, y_n)$$

To avoid making mistakes, one may check that the following conditions must be satisfied in every step.

1. The rightmost number in each row is always non-negative. This is guaranteed by the choice of the pivoting entry.

2. The value of the number in the lower right corner is always equal to the sum of those entries in the lower row which associate with x_i's at the top row (and similarly equal to the sum of those entries at the rightmost column associate with y_j's at the leftmost column.)

3. The value of the number in the lower right corner never increases.

Finally, one may also check that the result should satisfy the following two conditions.

1. Every entry of $\mathbf{p}A$ is larger than or equal to ν.

2. Every entry of $A\mathbf{q}^T$ is less than or equal to ν.

Example 3.3.6. *Solve the two-person zero sum game with game matrix*

$$\begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

Solution. Add $k = 1$ to each of the entries, we obtain the matrix

$$\begin{pmatrix} 0 & 6 & 4 & 3 \\ 7 & 0 & 1 & 5 \end{pmatrix}$$

Applying simplex algorithm, we have

	y_1	y_2	y_3	y_4	-1
x_1	0	6	4	3	1
x_2	7^*	0	1	5	1
-1	1	1	1	1	0

$\longrightarrow$

	x_2	y_2	y_3	y_4	-1
x_1	0	6^*	4	3	1
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	1	$\frac{6}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$

$\longrightarrow$

	x_2	x_1	y_3	y_4	-1
y_2	0	$\frac{1}{6}$	$\frac{2}{3}^*$	$\frac{1}{2}$	$\frac{1}{6}$
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	$-\frac{1}{6}$	$\frac{4}{21}$	$-\frac{3}{14}$	$-\frac{13}{42}$

$\longrightarrow$

	x_2	x_1	y_2	y_4	-1
y_3	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$
y_1	$-\frac{1}{7}$	$-\frac{1}{28}$	$-\frac{3}{14}$	$\frac{17}{28}$	$\frac{3}{28}$
-1	$-\frac{1}{7}$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{5}{14}$

The independent variables are $x_3, x_5, y_2, y_4, y_5, y_6$ and the basic variables are $x_1, x_2, x_4, x_6, y_1, y_3$. The basic solution is

$$x_3 = x_5 = 0, x_1 = \frac{3}{14}, x_2 = \frac{1}{7}, x_4 = \frac{2}{7}, x_6 = \frac{5}{14}$$
$$y_2 = y_4 = y_5 = y_6 = 0, y_1 = \frac{3}{28}, y_3 = \frac{1}{4}$$

The optimal value is $d = \dfrac{5}{14}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2) = \frac{14}{5}\left(\frac{3}{14}, \frac{1}{7}\right) = \left(\frac{3}{5}, \frac{2}{5}\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3, y_4) = \frac{14}{5}\left(\frac{3}{28}, 0, \frac{1}{4}, 0\right) = \left(\frac{3}{10}, 0, \frac{7}{10}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{14}{5} - 1 = \frac{9}{5}$$

□

Example 3.3.7. *Solve the two-person zero sum game with game matrix*

$$A = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

Solution. Add 2 to each of the entries, we obtain the matrix

$$\begin{pmatrix} 4 & 1 & 8 \\ 2 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$

Applying simplex method, we have

	y_1	y_2	y_3	-1
x_1	4^*	1	8	1
x_2	2	3	1	1
x_3	0	4	3	1
-1	1	1	1	0

$\longrightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{4}$	$\frac{1}{4}$	2	$\frac{1}{4}$
x_2	$-\frac{1}{2}$	$\frac{5}{2}^*$	-3	$\frac{1}{2}$
x_3	0	4	3	1
-1	$-\frac{1}{4}$	$\frac{3}{4}$	-1	$-\frac{1}{4}$

$\longrightarrow$

	x_1	x_2	y_3	-1
y_1	$\frac{3}{10}$	$-\frac{1}{10}$	$\frac{23}{10}$	$\frac{1}{5}$
y_2	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{6}{5}$	$\frac{1}{5}$
x_3	$\frac{4}{5}$	$-\frac{8}{5}$	$\frac{39}{5}$	$\frac{1}{5}$
-1	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{2}{5}$

The independent variables are $x_3, x_4, x_5, y_3, y_4, y_5$ and the basic variables are $x_1, x_2, x_6, y_1, y_2, y_6$. The basic solution is

$$\begin{aligned} & x_3 = x_4 = x_5 = 0, x_1 = \frac{1}{10}, x_2 = \frac{3}{10}, x_6 = \frac{1}{10} \\ & y_3 = y_4 = y_5 = 0, y_1 = \frac{1}{5}, y_2 = \frac{1}{5}, y_6 = \frac{1}{5} \end{aligned}$$

The optimal value is $d = \frac{2}{5}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{5}{2}\left(\frac{1}{10}, \frac{3}{10}, 0\right) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{5}{2}\left(\frac{1}{5}, \frac{1}{5}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{5}{2} - 1 = \frac{1}{2}$$

One may check the result by the following calculations

$$\begin{aligned} \mathbf{p}A &= \left(\frac{1}{4}, \frac{3}{4}, 0\right)\begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix} = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right) \\ A\mathbf{q}^T &= \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \end{aligned}$$

One sees that the row player may guarantee that his payoff is at least $\frac{1}{2}$ by using $\mathbf{p} = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$ and the column player may guarantee that the payoff to the row player is at most $\frac{1}{2}$ by using $\mathbf{q} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. □

3.4 Minimax theorem

In this section, we prove the minimax theorem (Theorem 2.1.10). The theorem was first published by John von Neumann in 1928. Another way to state the minimax theorem is that the row value and the column value of a matrix are always the same.

Definition 3.4.1 (Row and column values)**.** *Let A be an $m \times n$ matrix.*

1. *The* **row value** *of A is defined*[1] *by*

$$\nu_r(A) = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T$$

2. *The* **column value** *of A is defined by*

$$\nu_c(A) = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T$$

The row value $\nu_r(A)$ of a game matrix A is the largest payoff of the row player that he may guarantee himself. The column value $\nu_c(A)$ of A is the least payoff that the column player may guarantee that the row player cannot surpass. The strategies for the players to achieve these goals are called maximin and minimax strategies.

Definition 3.4.2 (Maximin and minimax strategies)**.** *Let A be an $m \times n$ matrix.*

1. *A* **maximin strategy** *is a strategy $\mathbf{p} \in \mathcal{P}^m$ for the row player such that*

$$\min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu_r(A)$$

2. *A* **minimax strategy** *is a strategy $\mathbf{q} \in \mathcal{P}^n$ for the column player such that*

$$\max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T = \nu_c(A)$$

It can be seen readily that we always have $\nu_r(A) \leq \nu_c(A)$ for any matrix A and we give a rigorous proof here.

---

[1] Note that since the payoff function $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T$ is continuous and the sets $\mathcal{P}^m, \mathcal{P}^n$ are compact, that is closed and bounded, the payoff function attains its maximum and minimum by extreme value theorem.

Theorem 3.4.3. *For any $m \times n$ matrix A, we have*

$$\nu_r(A) \leq \nu_c(A)$$

Proof. Let $\mathbf{p} \in \mathcal{P}^m$ be a maximin strategy for the row player and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy for the column player. Then we have

$$\begin{aligned} \nu_r(A) &= \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \\ &= \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T \\ &\leq \mathbf{p}A\mathbf{q}^T \\ &\leq \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T \\ &= \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T \\ &= \nu_c(A) \end{aligned}$$

□

Before we prove the minimax theorem, let's study some properties of convex sets.

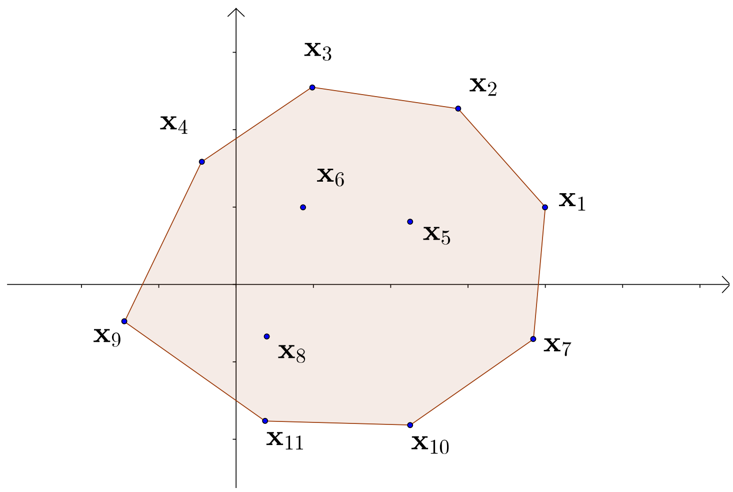
Definition 3.4.4 (Convex set)**.** *A set $C \subset \mathbb{R}^n$ is said to be* **convex** *if*

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C \text{ for any } \mathbf{x}, \mathbf{y} \in C,\ 0 \leq \lambda \leq 1$$

Geometrically, a set $C \subset \mathbb{R}^n$ is convex if the line segment joining any two points in C is contained in C. It is easy to see from the definition that intersection of convex sets is convex.

Definition 3.4.5 (Convex hull)**.** *The* **convex hull** *of a set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is defined by*

$$\begin{aligned} &\mathrm{Conv}(\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}) \\ = \ &\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \text{ with } \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{k} \lambda_i = 1\} \end{aligned}$$



The convex hull of a set of vectors can also be defined as the smallest convex set which contains all vectors in the set.

To prove the minimax theorem, we prove a lemma concerning properties of convex sets. Recall that the standard inner product and the norm on $\mathbb{R}^n$ are defined as follows. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$
2. $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

The following lemma says that we can always use a plane to separate the origin and a closed convex set C not containing the origin. It is a special case of the **hyperplane separation theorem**[2].

Lemma 3.4.6. *Let $C \subset \mathbb{R}^n$ be a closed convex set with $\mathbf{0} \notin C$. Then there exists $\mathbf{z} \in C$ such that*

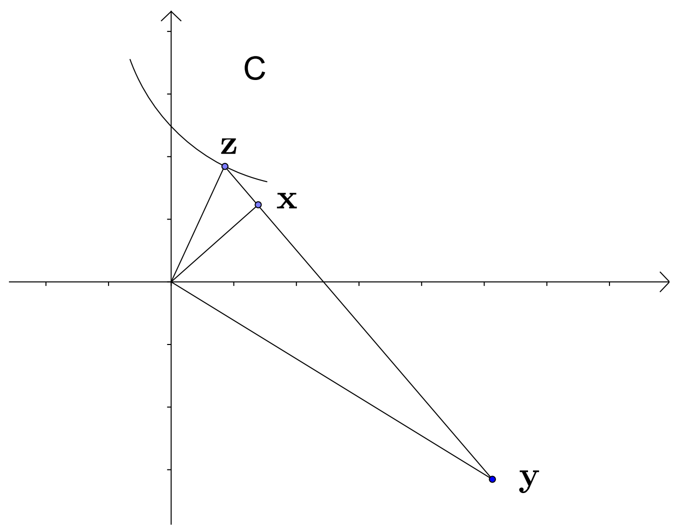
$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \textit{ for any } \mathbf{y} \in C$$

[2] The hyperplane separation theorem says that we can always use a hyperplane to separate two given sets which are closed and convex, and at least one of them is bounded.

Proof. Since C is closed, there exists $\mathbf{z} \in C$ such that

$$\|\mathbf{z}\| = \min_{\mathbf{y} \in C} \|\mathbf{y}\|$$

We are going to prove that $\langle \mathbf{z}, \mathbf{y} \rangle > 0$ for any $\mathbf{y} \in C$ by contradiction. Suppose there exists $\mathbf{y} \in C$ such that $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$. Let $\mathbf{x} \in \mathbb{R}^n$ be a point which lies on the straight line passing through $\mathbf{z}$, $\mathbf{y}$, and is orthogonal to $\mathbf{z} - \mathbf{y}$. The point $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$, that is lying between $\mathbf{z}$ and $\mathbf{y}$, because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$.



Since $\mathbf{z}, \mathbf{y} \in C$ and C is convex, we have $\mathbf{x} \in C$. (The expression for $\mathbf{x}$ is not important in the proof but let's include here for reference

$$\mathbf{x} = \frac{\langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{z} + \frac{\langle \mathbf{z} - \mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{y}$$

Note that $\frac{\langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2}, \frac{\langle \mathbf{z} - \mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \geq 0$ because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$ and $\frac{\langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} + \frac{\langle \mathbf{z} - \mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} = 1$ which shows that $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$.)

Moreover, we have

$$\begin{aligned} \|\mathbf{z}\|^2 &= \|\mathbf{x} + (\mathbf{z} - \mathbf{x})\|^2 \\ &= \|\mathbf{x}\|^2 + \|(\mathbf{z} - \mathbf{x})\|^2 \text{ (since } \mathbf{x} \perp \mathbf{z} - \mathbf{x}) \\ &> \|\mathbf{x}\|^2 \end{aligned}$$

which contradicts that $\mathbf{z}$ is a point in C closest to the origin $\mathbf{0}$. □

The following theorem says that for any matrix A, we have either $\nu_r(A) > 0$ or $\nu_c(A) \leq 0$. The key of the proof is to consider the convex hull C generated by the column vectors of A and the standard basis for $\mathbb{R}^m$, and study the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Theorem 3.4.7. *Let A be an $m \times n$ matrix. Then one of the following statements holds.*

1. *There exists probability vector $\mathbf{x} \in \mathcal{P}^m$ such that $\mathbf{x}A > \mathbf{0}$, that is all coordinates of $\mathbf{x}A$ are positive. In this case, $\nu_r(A) > 0$.*

2. *There exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T \leq \mathbf{0}$, that is all coordinates of $A\mathbf{y}^T$ are non-positive. In this case, $\nu_c(A) \leq 0$.*

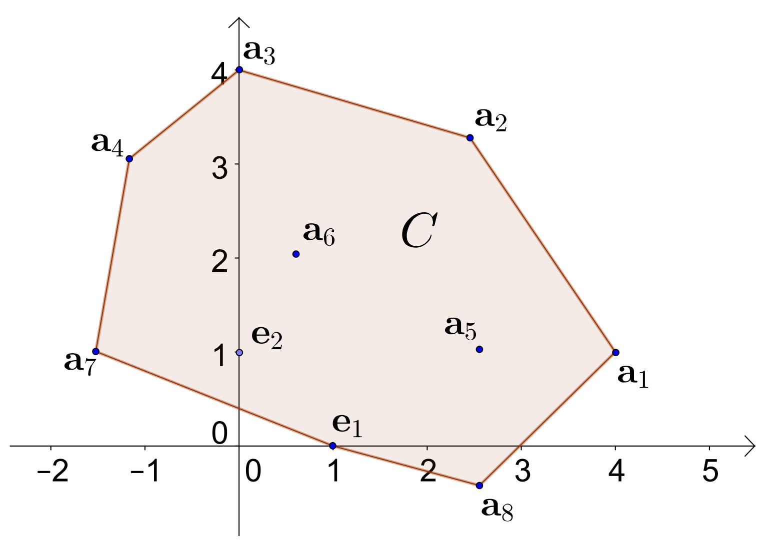
Proof. For $j = 1, 2, \cdots, n$, let

$$\mathbf{a}_j = (a_{1j}, a_{2j}, \cdots, a_{mj}) \in \mathbb{R}^m$$

In other words, $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$ are the column vectors of A and we may write $A = [\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T]$. Let

$$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

be the convex hull of $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$.



We are going to prove that the two statements in the theorem correspond to the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Case 1. Suppose $\mathbf{0} \notin C$. Then by Lemma 3.4.6, there exists $\mathbf{z} = (z_1, z_2, \cdots, z_m) \in \mathbb{R}^m$ such that

$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \text{ for any } \mathbf{y} \in C$$

In particular, we have

$$\langle \mathbf{z}, \mathbf{e}_i \rangle = z_i > 0 \text{ for any } i = 1, 2, \cdots, m$$

Then we may take

$$\mathbf{x} = \frac{\mathbf{z}}{z_1 + z_2 + \cdots + z_m} \in \mathcal{P}^m$$

and we have

$$\langle \mathbf{x}, \mathbf{a}_j \rangle = \frac{\langle \mathbf{z}, \mathbf{a}_j \rangle}{z_1 + z_2 + \cdots + z_m} > 0 \text{ for any } j = 1, 2, \cdots, n$$

which means $\mathbf{x}A > \mathbf{0}$. Let $\alpha > 0$ be the smallest coordinate of the vector $\mathbf{x}A$ and we have

$$\nu_r(A) \geq \min_{\mathbf{y} \in \mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \geq \alpha > 0$$

Case 2. Suppose $\mathbf{0} \in C$. Then there exists $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ with $\lambda_i \geq 0$ for all i, and $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$ such that

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n + \lambda_{n+1} \mathbf{e}_1 + \lambda_{n+2} \mathbf{e}_2 + \cdots + \lambda_{n+m} \mathbf{e}_m = \mathbf{0}$$

which implies

$$\begin{aligned} & A\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)^T \\ = \; & \lambda_1 \mathbf{a}_1^T + \lambda_2 \mathbf{a}_2^T + \cdots + \lambda_n \mathbf{a}_n^T \\ = \; & -(\lambda_{n+1} \mathbf{e}_1^T + \lambda_{n+1} \mathbf{e}_2^T + \cdots + \lambda_{n+m} \mathbf{e}_m^T) \\ = \; & -\left(\lambda_{n+1}, \lambda_{n+2}, \cdots, \lambda_{n+m}\right)^T \end{aligned}$$

Since $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ are linearly independent, at least one of $\lambda_1, \lambda_2, \cdots, \lambda_n$ is positive for otherwise all $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ are zero which contradicts $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$. Then we may take

$$\mathbf{y} = \frac{(\lambda_1, \lambda_2, \cdots, \lambda_n)}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \in \mathcal{P}^n$$

and we have

$$A\mathbf{y}^T = -\frac{1}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \begin{pmatrix} \lambda_{n+1} \\ \vdots \\ \lambda_{n+m} \end{pmatrix} \leq \mathbf{0}$$

which implies

$$v_c(A) \leq \max_{\mathbf{x} \in \mathcal{P}^m} \mathbf{x} A \mathbf{y}^T \leq 0$$

□

Now we give the proof of the minimax theorem (Theorem 2.1.10) which can be stated in the following form.

Theorem 3.4.8 (Minimax theorem)**.** *For any matrix A, the row value and columns value of A are equal. In other words, we have*

$$\nu_r(A) = \nu_c(A)$$

Proof. It has been proved that $\nu_r(A) \leq \nu_c(A)$ for any matrix A (Theorem 3.4.3). We are going to prove that $\nu_c(A) \leq \nu_r(A)$ by contradiction. Suppose there exists matrix A such that $\nu_r(A) < \nu_c(A)$. Let k be a real number such that $\nu_r(A) < k < \nu_c(A)$. Let A' be the matrix obtained by subtracting every entry of A by k. Then $\nu_r(A') = \nu_r(A) - k < 0$ and $\nu_c(A') = \nu_c(A) - k > 0$ which is impossible by applying Theorem 3.4.7 to A'. The contradiction shows that $\nu_c(A) \leq \nu_r(A)$ for any matrix A and the proof of the minimax theorem is complete. □

Exercise 3

1. Solve the following primal problems. Then write down the dual problems and the solutions to the dual problems.

   (a)
$$\begin{array}{rl} \max & f = 3y_1 + 5y_2 + 4y_3 + 12 \\ \text{subject to} & 3y_1 + 2y_2 + 2y_3 \leq 15 \\ & 4y_2 + 5y_3 \leq 24 \end{array}$$

   (b)
$$\begin{array}{rl} \max & f = 2y_1 + 4y_2 + 3y_3 + y_4 \\ \text{subject to} & 3y_1 + y_2 + y_3 + 4y_4 \leq 12 \\ & y_1 - 3y_2 + 2y_3 + 3y_4 \leq 7 \\ & 2y_1 + y_2 + 3y_3 - y_4 \leq 10 \end{array}$$

2. Solve the zero sum games with the following game matrices, that is find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

   (a) $\begin{pmatrix} 2 & -3 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 6 \\ -2 & -1 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 0 & -2 \\ -1 & -3 & 3 \\ -2 & 2 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 1 & -2 & 2 \\ -1 & 1 & -2 \end{pmatrix}$

   (f) $\begin{pmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}$

3. Prove that if C_1 and C_2 are convex sets in $\mathbb{R}^n$, then the following sets are also convex.

   (a) $C_1 \cap C_2$

   (b) $C_1 + C_2 = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}$

4. Let A be an $m \times n$ matrix. Prove that the set of maximin strategies for the row player of A is convex.

5. Let C be a convex set in $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in C$. Let $\mathbf{z} \in \mathbb{R}^n$ be a point on the straight line joining $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{z}$ is orthogonal to $\mathbf{x} - \mathbf{y}$.

   (a) Find $\mathbf{z}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

   (b) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle < 0$. Prove that $\mathbf{z} \in C$.

6. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$. Let $\nu_c(A)$ be the column value of A and let

$$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

   where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$. Prove that if $\nu_c(A) \leq 0$, then $\mathbf{0} \in C$.

4 Bimatrix games

4.1 Bimatrix games

In this chapter, we study bimatrix game. A bimatrix game is a two-person non-cooperative game with perfect information. In a bimatrix game, two players, player I and player II, choose their strategies simultaneously. Then the payoffs to the the players depend on the strategies used by the players. Unlike zero sum game, we have no assumption on the sum of payoffs to the players. A bimatrix game can be represented by two matrices, hence its name.

Definition 4.1.1 (Bimatrix game)**.** *The normal form of a* **bimatrix game** *is given by a pair of $m \times n$ matrices (A, B). The matrices A and B are payoff matrices for the row player (player I) and the column player (player II) respectively. Suppose the row player uses strategy $\mathbf{x} \in \mathcal{P}^m$ and the column player uses strategy $\mathbf{y} \in \mathcal{P}^n$. Then the payoff to the row player and column player are given by the payoff functions*

$$\begin{aligned} \pi(\mathbf{x}, \mathbf{y}) &= \mathbf{x}A\mathbf{y}^T \\ \rho(\mathbf{x}, \mathbf{y}) &= \mathbf{x}B\mathbf{y}^T \end{aligned}$$

respectively.

Definition 4.1.2. *The* **safety level***, or* **security level***, of the row player is*

$$\mu = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu(A)$$

where $\nu(A)$ denotes the value of the matrix A when A is considered as the game matrix of a two-person zero sum game. The safety level of the column player is

$$\nu = \max_{\mathbf{y}\in\mathcal{P}^n} \min_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}B\mathbf{y}^T = \nu(B^T)$$

where $\nu(B^T)$ is the value of the transpose B^T of B.

Note that the value of a matrix is defined to be the maximum payoff that the row payoff may guarantee himself. The safety level of the column player of the bimatrix game (A, B) is the value ν_{B^T} of the transpose B^T of B, not the value of B.

Definition 4.1.3 (Nash equilibrium)**.** *Let* (A, B) *be a game bimatrix. We say that a pair of strategies* $(\mathbf{p}, \mathbf{q})$ *is an* **equilibrium pair***, or* **mixed Nash equilibrium***, or just* **Nash equilibrium***, for* (A, B) *if*

$$\mathbf{x}A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T \text{ for any } \mathbf{x} \in \mathcal{P}^m$$

and

$$\mathbf{p}B\mathbf{y}^T \leq \mathbf{p}B\mathbf{q}^T \text{ for any } \mathbf{y} \in \mathcal{P}^n$$

Example 4.1.4 (Prisoner dilemma)**.** *Let*

$$(A, B) = \begin{pmatrix} (-5, -5) & (-1, -10) \\ (-10, -1) & (-2, -2) \end{pmatrix}$$

The strategy pair $(\mathbf{p}, \mathbf{q}) = ((1, 0), (1, 0))$ is a Nash equilibrium. The Nash equilibrium is unique in this example. □

Example 4.1.5 (Dating game)**.** *Consider*

$$(A, B) = \begin{pmatrix} (4, 2) & (0, 0) \\ (0, 0) & (1, 3) \end{pmatrix}$$

It is not difficult to see that the strategy pairs $(\mathbf{p}, \mathbf{q}) = ((1, 0), (1, 0))$ and $((0, 1), (0, 1))$ are Nash equilibria. The game has one more mixed Nash equilibrium. To find it, suppose the row player uses strategy $\mathbf{x} = (x, 1 - x)$, where $0 < x < 1$. Then

$$\mathbf{x}B = (x, 1 - x) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (2x, 3 - 3x)$$

It means that the payoff to the column player is $2x$, or $3 - 3x$ if the column player constantly uses his 1st, or 2nd strategies respectively. Setting $2x = 3 - 3x$, we have $x = 0.6$ and

$$(0.6, 0.4)B = (0.6, 0.4) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (1.2, 1.2)$$

Thus if the row player uses mixed strategy $(0.6, 0.4)$, then the payoff to the column player is always 1.2 no matter how the column player plays. Similarly suppose the column player uses $\mathbf{y} = (y, 1 - y)$, $0 \leq y \leq 1$. Then

$$A\mathbf{y}^T = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1 - y \end{pmatrix} = \begin{pmatrix} 4y \\ 1 - y \end{pmatrix}$$

It means that the payoff to the row player is $4y$, or $1-y$ if the row player constantly uses his 1st, or 2nd strategies respectively. Setting $4y = 1-y$, we have $y = 0.2$. Then

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}$$

Thus if the column player uses mixed strategy $(0.2, 0.8)$, then the payoff to the row player is always 0.8 no matter how the row player plays. Therefore the strategy pair $(\mathbf{p}, \mathbf{q}) = ((0.6, 0.4), (0.2, 0.8))$ is a Nash equilibrium. In conclusion, the dating game has three Nash equilibria and we list them in the following table.

Nash equilibrium and the corresponding payoff pair

Row player's strategy $\mathbf{p}$	Column player's strategy $\mathbf{q}$	Payoff pair (π, ρ)
$(1, 0)$	$(1, 0)$	$(4, 2)$
$(0, 1)$	$(0, 1)$	$(1, 3)$
$(0.6, 0.4)$	$(0.2, 0.8)$	$(0.8, 1.2)$

□

Note that in the third Nash equilibrium of the above example, the strategy for the row player $\mathbf{p} = (0.6, 0.4)$ is the minimax strategy for the column player of B^T, not the maximin strategy for the row player of A. That means what the row player should do is to fix the payoff to its opponent (the column player) to be 1.2 instead of guaranteeing the payoff to himself to be 0.8. Similarly, the strategy for the column player $\mathbf{q} = (0.2, 0.8)$ in this Nash equilibrium is the minimax strategy for the column player of A. So the column player should use a strategy to fix the row player's payoff instead of guaranteeing his own payoff.

4.2 Nash's theorem

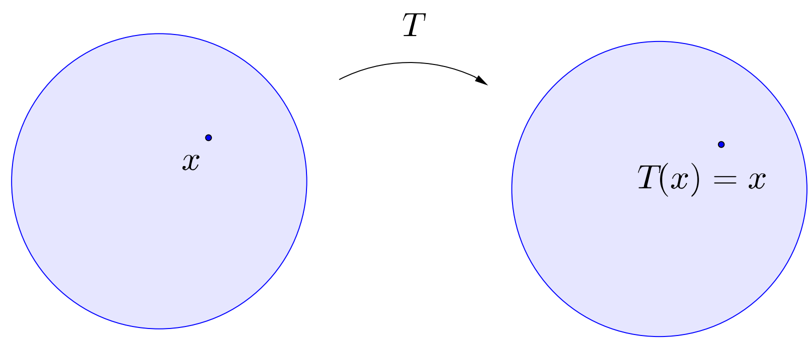
One of the most fundamental works in game theory is the following theorem of Nash which greatly extended the minimax theorem (Theorem 2.1.10).

Theorem 4.2.1 (Nash's theorem)**.** *Every finite*[3] *game with finite number of players has at least one Nash equilibrium.*

[3] A game is finite if the number of strategies of each player is finite.

Nash invoked the following celebrated theorem in topology to prove his theorem.

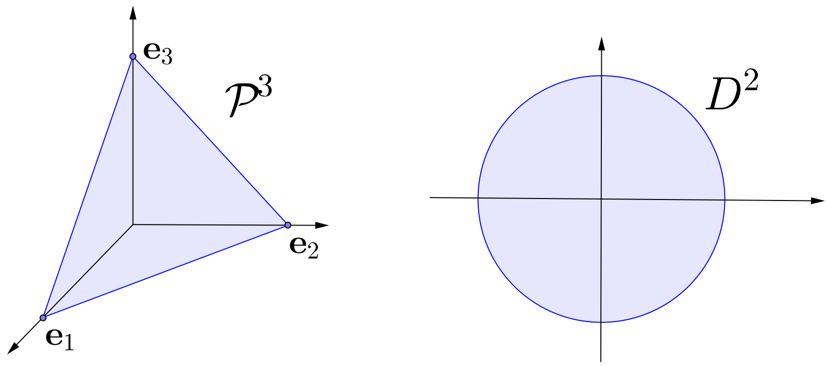
Theorem 4.2.2 (Brouwer's fixed-point theorem)**.** *Let X be a topological space which is homeomorphic to the closed unit ball $D^n = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| \leq 1\}$. Then any continuous map $T : X \to X$ has at least one fixed-point, that is, there exists $x \in X$ such that $T(x) = x$.*



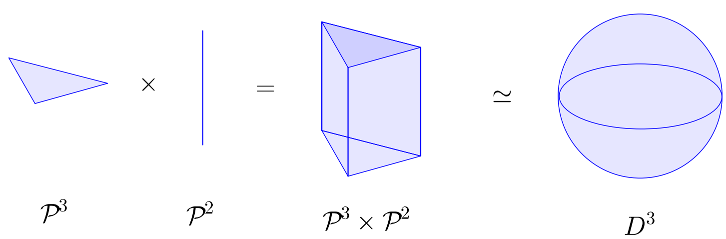
Remarks:

1. Two topological space X and Y are homeomorphic if there exists bijective map $\varphi : X \to Y$ such that both φ and its inverse φ^{-1} are continuous.

2. The set $\mathcal{P}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1, \cdots, x_n \geq 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}$ of probability vectors in $\mathbb{R}^n$ is homeomorphic to D^{n-1}.



Moreover $\mathcal{P}^m \times \mathcal{P}^n$ is homeomorphic to D^{m+n-2}.



The proof of the Brouwer's fixed-point theorem is out of the propose and scope of this notes. Now we give the proof of Nash's theorem assuming the Brouwer's fixed-point theorem.

Proof of Nash's theorem. For simplicity, we consider two-person game only. The proof for the general case is similar. Let (A, B) be the game bimatrix of a two-person game. Define $T : \mathcal{P}^m \times \mathcal{P}^n \to \mathcal{P}^m \times \mathcal{P}^n$ by

$$T(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}) = ((u_1, u_2, \cdots, u_m), (v_1, v_2, \cdots, v_n))$$

where for $k = 1, 2, \cdots, m$ and $l = 1, 2, \cdots, n$,

$$u_k = \frac{x_k + c_k}{1 + \sum_{i=1}^{m} c_i} \text{ and } v_l = \frac{y_l + d_l}{1 + \sum_{j=1}^{n} d_j}$$

and

$$c_k = \max\{\mathbf{e}_k A\mathbf{y}^T - \mathbf{x}A\mathbf{y}^T, 0\} \text{ and } d_l = \max\{\mathbf{x}B\mathbf{e}_l^T - \mathbf{x}B\mathbf{y}^T, 0\}$$

Here $\mathbf{e}_k$, $\mathbf{e}_l$ are vectors in the standard bases in $\mathbb{R}^m$, $\mathbb{R}^n$ respectively. Note that $\mathbf{u} \in \mathcal{P}^m$ and $\mathbf{v} \in \mathcal{P}^n$ because

$$c_k, d_l \geq 0$$

and

$$\begin{aligned} \sum_{k=1}^{m} \left(\frac{x_k + c_k}{1 + \sum_{i=1}^{m} c_i} \right) &= \frac{\sum_{k=1}^{m} x_k + \sum_{k=1}^{m} c_k}{1 + \sum_{i=1}^{m} c_i} = \frac{1 + \sum_{k=1}^{m} c_k}{1 + \sum_{i=1}^{m} c_i} = 1 \\ \sum_{l=1}^{n} \left(\frac{y_l + d_l}{1 + \sum_{j=1}^{n} d_j} \right) &= \frac{\sum_{l=1}^{n} y_l + \sum_{l=1}^{n} d_l}{1 + \sum_{j=1}^{n} d_j} = \frac{1 + \sum_{l=1}^{n} d_l}{1 + \sum_{j=1}^{n} d_j} = 1 \end{aligned}$$

Now T is a continuous map from $\mathcal{P}^m \times \mathcal{P}^n$ to $\mathcal{P}^m \times \mathcal{P}^n$. By Brouwer's fixed-point theorem (Theorem 4.2.2), there exists $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}^m \times \mathcal{P}^n$ such that

$$T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$$

The proof of Nash's theorem is complete if we can prove that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose on the contrary that $(\mathbf{p}, \mathbf{q})$ is not a Nash equilibrium. Then either there exists $\mathbf{r} \in \mathcal{P}^m$ such that $\mathbf{r}A\mathbf{q}^T > \mathbf{p}A\mathbf{q}^T$ or there exists $\mathbf{s} \in \mathcal{P}^n$ such that $\mathbf{p}B\mathbf{s}^T > \mathbf{p}B\mathbf{q}^T$. Without loss of generality, we consider the former case. Write $\mathbf{r} = (r_1, r_2, \cdots, r_m)$. Since

$$\mathbf{p}A\mathbf{q}^T < \mathbf{r}A\mathbf{q}^T = \sum_{k=1}^{m} r_k \mathbf{e}_k A\mathbf{q}^T$$

and $\mathbf{r}$ is a probability vector, we see that there exists $1 \leq k \leq m$ such that

$$\mathbf{p}A\mathbf{q}^T < \mathbf{e}_k A\mathbf{q}^T$$

It follows that

$$c_k = \max\{\mathbf{e}_k A\mathbf{q}^T - \mathbf{p}A\mathbf{q}^T, 0\} > 0$$

and thus $\sum_{i=1}^{m} c_i > 0$. On the other hand, since

$$\mathbf{p}A\mathbf{q}^T = \sum_{i=1}^{m} p_i \mathbf{e}_i A\mathbf{q}^T$$

and $\mathbf{p}$ is a probability vector, there exists $1 \leq r \leq m$ such that $p_r > 0$ and

$$\mathbf{e}_r A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$$

which implies, by the definition of c_r, that $c_r = 0$. Hence we have

$$\frac{p_r + c_r}{1 + \sum_{i=1}^{m} c_i} = \frac{p_r}{1 + \sum_{i=1}^{m} c_i} \leq \frac{p_r}{1 + c_k} < p_r$$

which contradicts that $(\mathbf{p}, \mathbf{q})$ is a fixed-point of T. Therefore $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium and the proof of Nash's theorem is complete. □

We have seen in the proof of Nash's theorem that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium if it is a fixed-point of T. As a matter of fact, the converse of this statement is also true. For if $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, then $\mathbf{e}_i A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$ for any $1 \leq i \leq m$. Thus $c_i = 0$ for any $1 \leq i \leq m$. Similarly $d_j = 0$ for any $1 \leq j \leq n$. Therefore $T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$.

To find Nash equilibria of a 2×2 game bimatrix (A, B), we may let $\mathbf{x} = (x, 1-x)$, $\mathbf{y} = (y, 1-y)$ and consider the payoff functions

$$\begin{aligned} \pi(x, y) &= \pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T \\ \rho(x, y) &= \rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}B\mathbf{y}^T \end{aligned}$$

Define

$$\begin{aligned} P &= \{(x, y) : \pi(x, y) \text{ attains its maximum at } x \text{ for fixed } y.\} \\ Q &= \{(x, y) : \rho(x, y) \text{ attains its maximum at } y \text{ for fixed } x.\} \end{aligned}$$

Then $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if $(x, y) \in P \cap Q$.

Example 4.2.3 (Prisoner dilemma). *Consider the prisoner dilemma (Example 4.1.4) with bimatrix*

$$(A, B) = \begin{pmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{pmatrix}$$

The payoff to the row player is given by

$$\begin{aligned} \pi(x,y) &= (x, 1-x)\begin{pmatrix} -5 & -1 \\ -10 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x)\begin{pmatrix} -4y-1 \\ -8y-2 \end{pmatrix} \end{aligned}$$

Since $-8y - 2 < -4y - 1$ for any $0 \leq y \leq 1$, we have

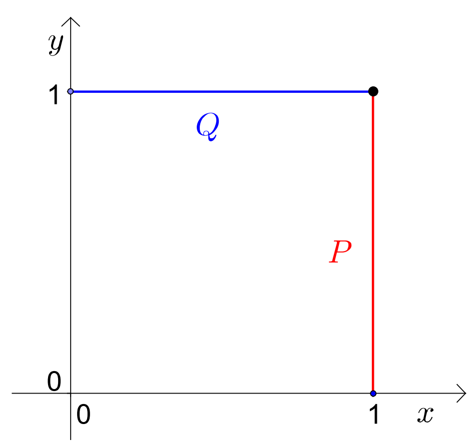
$$P = \{(1, y) : 0 \leq y \leq 1\}$$

On the other hand,

$$\begin{aligned} \rho(x,y) &= (x, 1-x)\begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (-4x-1, -8x-2)\begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Since $-8x - 2 < -4x - 1$ for any $0 \leq x \leq 1$, we have

$$Q = \{(x, 1) : 0 \leq p \leq 1\}$$



Now

$$P \cap Q = \{(1, 1)\}$$

Therefore the game has a unique Nash equilibrium $(\mathbf{p}, \mathbf{q}) = ((1, 0), (1, 0))$.$\square$

Example 4.2.4 (Dating game)**.** *Consider the dating game (Example 4.1.5) with bimatrix*

$$(A, B) = \begin{pmatrix} (4, 2) & (0, 0) \\ (0, 0) & (1, 3) \end{pmatrix}$$

We have

$$\begin{aligned} \pi(x, y) &= (x, 1-x) \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x) \begin{pmatrix} 4y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 4y < 1-y & \text{if} \quad 0 \leq y < \dfrac{1}{5} \\ 4y = 1-y & \text{if} \quad y = \dfrac{1}{5} \\ 4y > 1-y & \text{if} \quad \dfrac{1}{5} < y \leq 1 \end{cases}$$

Thus

$$P = \left\{(x, y) : \left(x = 0 \wedge 0 \leq y < \frac{1}{5}\right) \vee \left(0 \leq x \leq 1 \wedge y = \frac{1}{5}\right) \vee \left(x = 1 \wedge \frac{1}{5} < y \leq 1\right)\right\}$$

On the other hand,

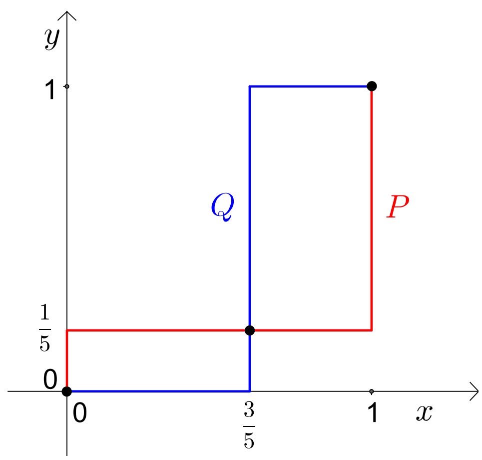
$$\begin{aligned} \rho(x, y) &= (x, 1-x) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (2x, 3-3x) \begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 2x < 3-3x & \text{if} \quad 0 \leq x < \dfrac{3}{5} \\ 2x = 3-3x & \text{if} \quad x = \dfrac{3}{5} \\ 2x > 3-3x & \text{if} \quad \dfrac{3}{5} < x \leq 1 \end{cases}$$

Thus

$$Q = \left\{ (x,y) : \left(0 \le x < \frac{3}{5} \wedge y = 0\right) \vee \left(x = \frac{3}{5} \wedge 0 \le y \le 1\right) \vee \left(\frac{3}{5} < x \le 1 \wedge y = 1\right) \right\}$$



Now

$$P \cap Q = \left\{ (0,0), (1,1), \left(\frac{3}{5}, \frac{1}{5}\right) \right\}$$

Therefore the game has three Nash equilibria which are listed together with the associated payoff pairs in the following table.

$\mathbf{p}$	$\mathbf{q}$	(π, ρ)
$(0,1)$	$(0,1)$	$(1,3)$
$(1,0)$	$(1,0)$	$(4,2)$
$\left(\frac{3}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{5}, \frac{4}{5}\right)$	$\left(\frac{4}{5}, \frac{6}{5}\right)$

□

Definition 4.2.5. *Let (A, B) be a game bimatrix.*

1. *We say that two Nash equilibria $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$ are* **interchangeable** *if $(\mathbf{p}', \mathbf{q})$ and $(\mathbf{p}, \mathbf{q}')$ are also Nash equilibria.*

2. *We say that two Nash equilibria* $(\mathbf{p},\mathbf{q})$ *and* $(\mathbf{p}',\mathbf{q}')$ *are equivalent if*

$$\pi((\mathbf{p},\mathbf{q}),\rho(\mathbf{p},\mathbf{q})) = \pi((\mathbf{p}',\mathbf{q}'),\rho(\mathbf{p}',\mathbf{q}'))$$

3. *We say that a bimatrix game* (A,B) *is* **solvable in the Nash sense** *if any two Nash equilibria are interchangeable and equivalent.*

For the prisoner dilemma (Example 4.2.3), there is only one Nash equilibrium. Thus the prisoner dilemma is solvable in the Nash sense. For the dating game (Example 4.2.4), there are three Nash equilibria which are not interchangeable. So the dating game is not solvable in the Nash sense.

Example 4.2.6. *Solve the game bimatrix*

$$(A,B) = \begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

Solution. Consider

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} -4y+5 \\ y+3 \end{pmatrix}$$

Now

$$\begin{cases} -4y+5 > y+3 & \text{if} \quad 0 \le y < \dfrac{2}{5} \\ -4y+5 = y+3 & \text{if} \quad y = \dfrac{2}{5} \\ -4y+5 < y+3 & \text{if} \quad \dfrac{2}{5} < y \le 1 \end{cases}$$

We see that

$$P = \left\{(x,y) : \left(x = 0 \wedge \frac{2}{5} < y \le 1\right) \vee \left(0 \le x \le 1 \wedge y = \frac{2}{5}\right) \vee \left(x = 1 \wedge 0 \le y < \frac{2}{5}\right)\right\}$$

On the other hand

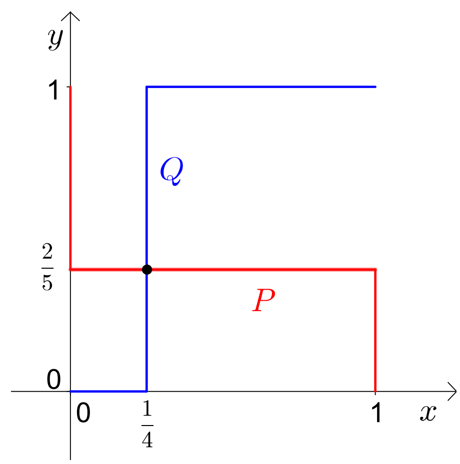
$$\mathbf{x}B = (x, 1-x)\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = (2x+2, -2x+3)$$

and

$$\begin{cases} 2x+2 < -2x+3 & \text{if} \quad 0 \le x < \dfrac{1}{4} \\ 2x+2 = -2x+3 & \text{if} \quad x = \dfrac{1}{4} \\ 2x+2 > -2x+3 & \text{if} \quad \dfrac{1}{4} < x \le 1 \end{cases}$$

We see that

$$Q = \left\{ (x,y) : \left(0 \le x < \frac{1}{4} \wedge y = 0 \right) \vee \left(x = \frac{1}{4} \wedge 0 \le y \le 1 \right) \vee \left(\frac{1}{4} < x \le 1 \wedge y = 1 \right) \right\}$$



Now

$$P \cap Q = \left\{ \left(\frac{1}{4}, \frac{2}{5} \right) \right\}$$

Therefore the game has Nash equilibrium

$$(\mathbf{p}, \mathbf{q}) = \left(\left(\frac{1}{4}, \frac{3}{4} \right), \left(\frac{2}{5}, \frac{3}{5} \right) \right)$$

and is solvable in the Nash sense since the Nash equilibrium is unique. □

4.3 Nash bargaining model

A bimatrix game can be played as a cooperative game with **non-transferable utility**. This means the players may make agreements on what strategies they are going to use. However they are not allowed to share the payoffs they obtained in the game. In such a game, players may use joint strategies.

Definition 4.3.1. *Let (A, B) be an $m \times n$ bimatrix of a two-person game.*

1. *A* **joint strategy** *of* (A, B) *is an* $m \times n$ *matrix*

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{pmatrix}$$

which satisfies

(i) $p_{ij} \geq 0$ *for any* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$

(ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

In other words, P *is a joint strategy if it is a probability matrix. The set of all* $m \times n$ *probability matrices is denoted by*

$$\mathcal{P}^{m\times n} = \{P = [p_{ij}] : p_{ij} \geq 0 \text{ and } \sum p_{ij} = 1\}$$

In particular, if $\mathbf{p} = (p_1, \cdots, p_m) \in \mathcal{P}^m$ *and* $\mathbf{q} = (q_1, \cdots, q_n) \in \mathcal{P}^n$, *then*

$$\mathbf{p}^T\mathbf{q} = \begin{pmatrix} p_1q_1 & \cdots & p_1q_n \\ \vdots & \ddots & \vdots \\ p_mq_1 & \cdots & p_mq_n \end{pmatrix} \in \mathcal{P}^{m\times n}$$

is a joint strategy. In this case, the row player uses strategy $\mathbf{p}$ *and the column player uses strategy* $\mathbf{q}$ *independently. Not all joint strategies are of this form. For example*

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

cannot be expressed as the form $\mathbf{p}^T\mathbf{q}$. *When this joint strategy is used, the players may flip a coin and both use their first strategies if a 'head' is obtained and both use their second strategies if a 'tail' is obtained.*

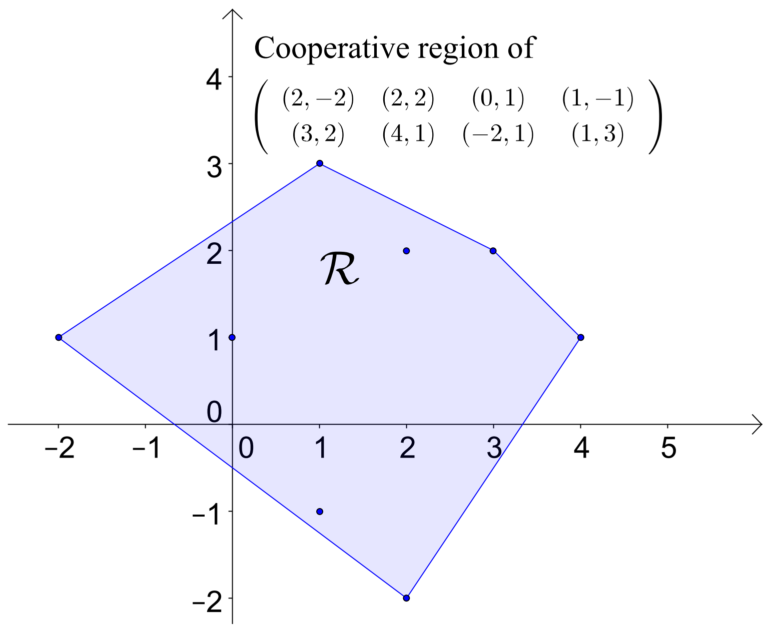
2. *For joint strategy* $P = [p_{ij}] \in \mathcal{P}^{m\times n}$, *the payoff* u *to the row player and the payoff* v *to the column player are given by the payoff pair*

$$\begin{aligned} (u(P), v(P)) &= \left(\sum_{i,j} a_{ij}p_{ij}, \sum_{i,j} b_{ij}p_{ij}\right) \\ &= \sum_{i,j} p_{ij}(a_{ij}, b_{ij}) \end{aligned}$$

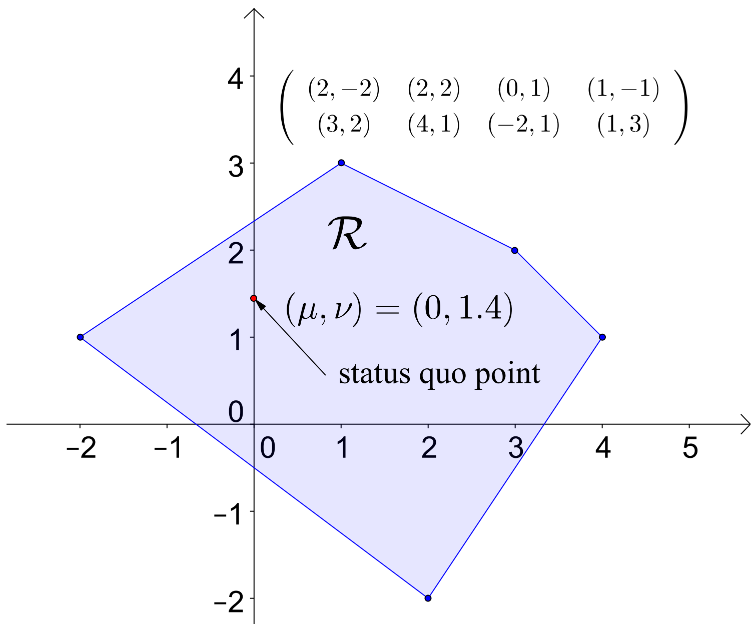
3. *The* **cooperative region** *of* (A, B) *is the set of all feasible payoff pairs*

$$\begin{aligned}\mathcal{R} &= \{(u(P), v(P)) \in \mathbb{R}^2 : P \in \mathcal{P}^{m\times n}\} \\ &= \left\{(u, v) \in \mathbb{R}^2 : (u, v) = \sum_{i,j} p_{ij}(a_{ij}, b_{ij}) \text{ for some } [p_{ij}] \in \mathcal{P}^{m\times n}\right\}\end{aligned}$$

In other words, the cooperative region $\mathcal{R}$ *is the convex hull of the set of points* $\{(a_{ij}, b_{ij}) : 1 \le i \le m, 1 \le j \le n\}$ *in* $\mathbb{R}^2$. *Note that* $\mathcal{R}$ *is a closed convex polygon in* $\mathbb{R}^2$.



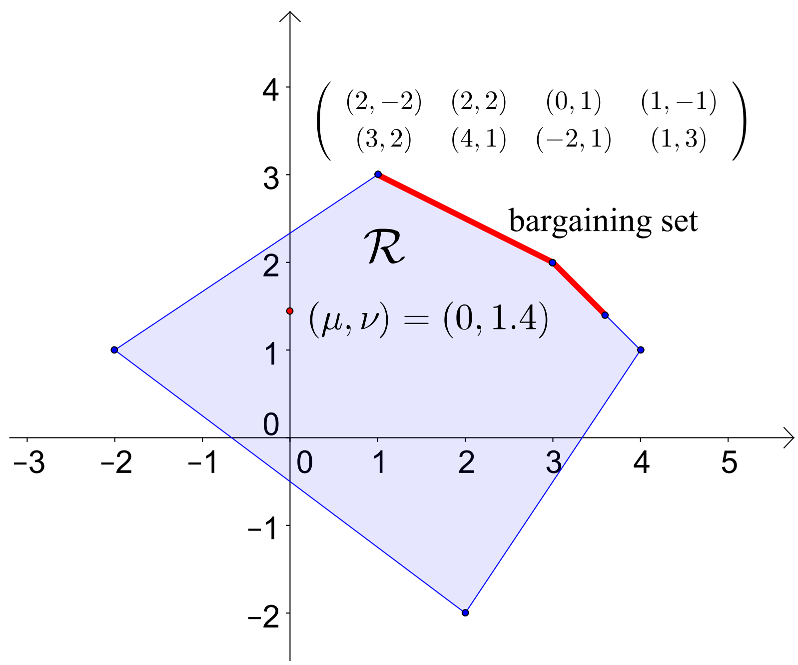
4. *The* **status quo point** *is the payoff pair* (μ, ν) *for the players associated to the solution of the game when* (A, B) *is considered as a non-cooperative game. In other words, the status quo point is the payoffs that the players may expect if the negotiations break down. Unless otherwise specified, we will take* $(\mu, \nu) = (\nu(A), \nu(B^T))$ *to be the status quo point where* $v(A)$ *and* $\nu(B^T)$ *are the values of* A *and the transpose* B^T *of* B *respectively.*



5. *We say that a payoff pair* (u, v) *is* **Pareto optimal** *if* $u' \geq u$, $v' \geq v$ *and* $(u', v') \in \mathcal{R}$ *implies* $(u', v') = (u, v)$ *where* $\mathcal{R}$ *is the cooperative region.*

6. *The* **bargaining set** *of* (A, B) *is the set of Pareto optimal payoff pairs* $(u, v) \in \mathcal{R}$ *such that* $u \geq \mu$ *and* $v \geq \nu$ *where* (μ, ν) *is the status quo point. In other words, the bargaining set is*

$$\{(u, v) \in \mathcal{R} : u \geq \mu, v \geq \nu \text{ and } (u, v) \text{ is Pareto optimal}\}$$



When the status quo point is not Pareto optimal, the two players of the game would have a tendency to cooperate. The **bargaining problem** is a problem to understand how the players should cooperate in this situation. Nash proposed that the solution to the bargaining problem is a function, called the arbitration function, depending only on the cooperative region $\mathcal{R}$ and the status quo point $(\mu, \nu) \in \mathcal{R}$, which satisfies certain properties called Nash bargaining axioms.

Definition 4.3.2 (Nash bargaining axioms)**.** *An* **arbitration function** *is a function* $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$ *defined for a closed and bounded convex set* $\mathcal{R} \subset \mathbb{R}^2$ *(cooperative region) and a point* $(\mu, \nu) \in \mathcal{R}$ *(status quo point) such that the following* **Nash bargaining axioms** *are satisfied.*

1. *(Individual rationality)* $\alpha \geq \mu$ *and* $\beta \geq \nu$*.*

2. *(Pareto optimality) For any* $(u, v) \in \mathcal{R}$*, if* $u \geq \alpha$ *and* $v \geq \nu$*, then* $(u, v) = (\alpha, \beta)$*.*

3. *(Feasibility)* $(\alpha, \beta) \in \mathcal{R}$*.*

4. *(Independence of irrelevant alternatives) If $\mathcal{R}' \subset \mathcal{R}$, $(\mu, \nu) \in \mathcal{R}'$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu)) \in \mathcal{R}'$, then $A(\mathcal{R}', (\mu, \nu)) = (\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$.*

5. *(Invariant under linear transformation) Let $a, b, c, d \in \mathbb{R}$ be any real numbers with $a, c > 0$. Let $\mathcal{R}' = \{(au + b, cv + d) : (u, v) \in \mathcal{R}\}$ and $(\mu', \nu') = (a\mu + b, c\nu + d)$. Then $A(\mathcal{R}', (\mu', \nu')) = (a\alpha + b, c\beta + d)$.*

6. *(Symmetry) Suppose $\mathcal{R}$ is symmetry, that is $(u, v) \in \mathcal{R}$ implies $(v, u) \in \mathcal{R}$, and $\mu = \nu$. Then $\alpha = \beta$.*

Theorem 4.3.3 (Nash bargaining solution)**.** *There exists a unique arbitration function $A(\mathcal{R}, (\mu, \nu))$ for closed and bounded convex set $\mathcal{R}$ and $(\mu, \nu) \in \mathcal{R}$ which satisfies the Nash bargaining axioms.*

Before proving Theorem 4.3.3, first we prove a lemma.

Lemma 4.3.4. *Let $\mathcal{R} \subset \mathbb{R}^2$ be any closed and bounded convex set and $(\mu, \nu) \in \mathcal{R}$. Let*

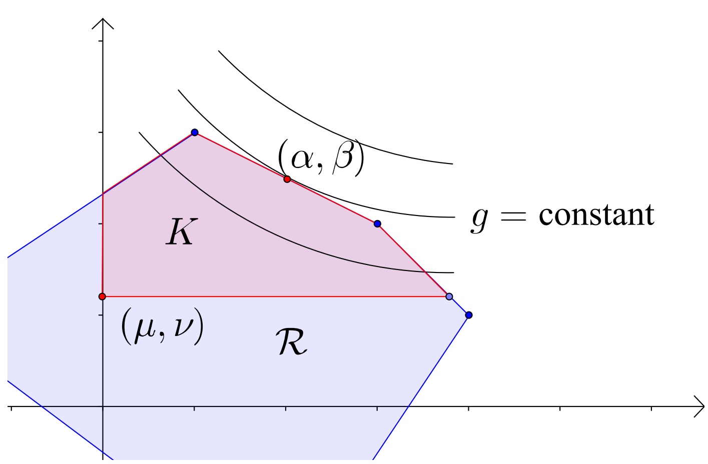
$$K = \{(u, v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}$$

Let $g : K \to \mathbb{R}$ be the function defined by

$$g(u, v) = (u - \mu)(v - \nu) \text{ for } (u, v) \in K$$

Suppose $U = \{(u, v) \in K : u > \mu, v > \nu\} \neq \emptyset$. Then there exists unique $(\alpha, \beta) \in K$ such that

$$g(\alpha, \beta) = \max_{(u,v) \in K} g(u, v)$$



Proof. Since g is continuous and K is closed and bounded, g attains its maximum at some point $(\alpha, \beta) \in K$ and let

$$M = \max_{(u,v)\in K} g(u, v)$$

be the maximum value of g on K. We are going to prove by contradiction that the maximum point of g on K is unique. Suppose on the contrary that there exists $(\alpha', \beta') \in K$ with $(\alpha', \beta') \neq (\alpha, \beta)$ such that

$$g(\alpha', \beta') = g(\alpha, \beta) = M$$

Then either $\alpha' > \alpha$ and $\beta' < \beta$, or $\alpha' < \alpha$ and $\beta' > \beta$. In both case we have $(\alpha - \alpha')(\beta' - \beta) > 0$. Observe that the mid-point $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ of (α, β) and (α', β') lies in K since K is convex. On the other hand, the value of g at $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ is

$$\begin{aligned}
& g\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right) \\
= & \left(\frac{\alpha+\alpha'}{2} - \mu, \frac{\beta+\beta'}{2} - \nu\right) \\
= & \frac{1}{4}\left((\alpha-\mu) + (\alpha'-\mu)\right)\left((\beta-\nu) + (\beta'-\nu)\right) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta'-\nu) \\
& +(\alpha'-\mu)(\beta-\nu) + (\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu) + (\alpha-\mu)((\beta'-\beta) + (\beta-\nu)) \\
& +(\alpha'-\mu)((\beta-\beta') + (\beta'-\nu)) + (\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}(2(\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta'-\beta) \\
& +(\alpha'-\mu)(\beta-\beta') + 2(\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}\left(2g(\alpha,\beta) + (\alpha-\alpha')(\beta'-\beta) + 2g(\alpha',\beta')\right) \\
= & \frac{1}{4}\left(2M + (\alpha-\alpha')(\beta'-\beta) + 2M\right) \\
> & M
\end{aligned}$$

This contradicts that the maximum value of g on K is M. Therefore g attains its maximum at a unique point. □

Proof of existence of arbitration function. For any closed and bounded convex set $\mathcal{R}$ and $(\mu,\nu) \in \mathcal{R}$, let $K = \{(u,v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}$, $U = \{(u,v) \in \mathcal{R} : u > \mu, v > \nu\}$ and define $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$ as follows:

1. If $U \neq \emptyset$, then $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu)) \in K$ is the unique maximum point of $g(u,v) = (u-\mu)(v-\nu)$ in K, that is
$$g(\alpha,\beta) = \max_{(u,v)\in K} g(u,v)$$
2. If $U = \emptyset$, then $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu)) \in K$ is the unique maximum point of $u+v$ on K, that is
$$\alpha+\beta = \max_{(u,v)\in K}(u+v)$$

We are going to prove that the function $A(\mathcal{R},(\mu,\nu))$ satisfies the Nash bargaining axioms. We prove only for the first case $U \neq \emptyset$ and the second case is obvious.

1. (Individual rationality) It follows by the definition that $(\alpha,\beta) \in K$ and we have $\alpha \geq \mu$ and $\beta \geq \nu$.
2. (Pareto optimality) Suppose there exists $(\alpha',\beta') \in \mathcal{R}$ such that $\alpha' \geq \alpha$ and $\beta' \geq \beta$. Then $g(\alpha',\beta') \geq g(\alpha,\beta)$ which implies that $(\alpha',\beta') = (\alpha,\beta)$ since the maximum point of g on K is unique.
3. (Feasibility) Since $(\alpha,\beta) \in K \subset \mathcal{R}$ by definition, we have $(\alpha,\beta) \in \mathcal{R}$.
4. (Independence of irrelevant alternatives) Suppose $\mathcal{R}' \subset \mathcal{R}$ is a subset of $\mathcal{R}$ which contains both (μ,ν) and (α,β). Since g attains its maximum at (α,β) on K, it also attains its maximum at (α,β) on $K' = K \cap \mathcal{R}'$. Thus
$$A(\mathcal{R}',(\mu,\nu)) = (\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$$
5. (Invariant under linear transformation) Let $a,b,c,d \in \mathbb{R}$ with $a,c > 0$. Let $\mathcal{R}' = \{(u',v') = (au+b, cv+d) : (u,v) \in \mathcal{R}\}$ and $(\mu',\nu') = (a\mu+b, c\nu+d)$. Then
$$\begin{aligned} g'(u',v') &= (u'-\mu')(v'-\nu') \\ &= ((au+b)-(a\mu+b))((cv+d)-(c\nu+d)) \\ &= ac(u-\mu)(v-\nu) \\ &= acg(u,v) \end{aligned}$$

Hence g' attains its maximum at $(\alpha', \beta') = (a\alpha + b, c\beta + d)$ on $K' = \{(u', v') = (au + b, cv + d) : (u, v) \in K\}$ since g attains its maximum at (α, β) on K. Therefore $A(\mathcal{R}', (\mu, \nu)) = (\alpha', \beta')$.

6. (Symmetry) Suppose $\mathcal{R}$ is symmetric and $\mu = \nu$. Then

$$g(u, v) = (u - \mu)(v - \mu) = g(v, u)$$

and $(v, u) \in K$ if and only if $(u, v) \in K$. Thus if g attains its maximum at (α, β) on K, then g also attains its maximum at (β, α) on K. By uniqueness of maximum point of g on K, we see that $(\beta, \alpha) = (\alpha, \beta)$ which implies $\alpha = \beta$.

□

Proof of uniqueness of arbitration function. Suppose $A'(\mathcal{R}, (\mu, \nu))$ is another arbitration function satisfying the Nash bargaining axioms. Let $\mathcal{R}$ be a closed and bounded convex set and $(\mu, \nu) \in \mathcal{R}$. By applying a linear transformation, we may assume that $(\mu, \nu) = (0, 0)$ and $(\alpha, \beta) = A(\mathcal{R}, (0, 0)) = (0, 0), (1, 0), (0, 1)$ or $(1, 1)$. We are going to prove that $A'(\mathcal{R}, (0, 0)) = (\alpha, \beta)$.

Case 1. $(\alpha, \beta) = (0, 0)$:

In this case $K = \{(0, 0)\}$ and we have $A'(\mathcal{R}, (0, 0))$ since $(\alpha, \beta) \in K$.

Case 2. $(\alpha, \beta) = (1, 0)$ or $(0, 1)$:

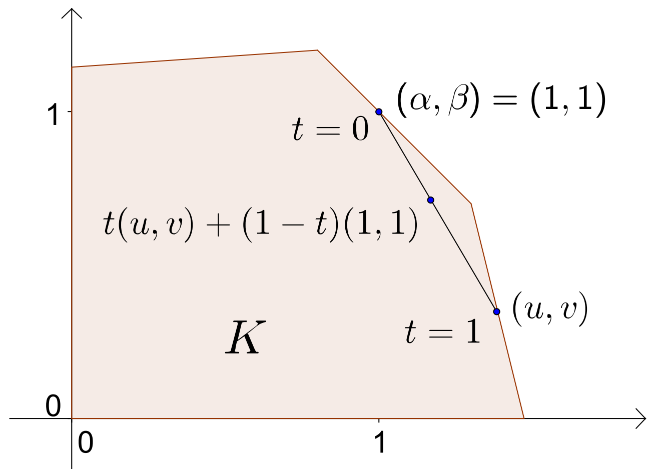
We consider the case for $(\alpha, \beta) = (1, 0)$ and the other case is similar. By definition of (α, β), we must have $K = \{(u, 0) : 0 \leq u \leq 1\}$. By the individual rationality, we have $A'(\mathcal{R}, (0, 0)) \in K$. By Pareto optimality, we have $A'(\mathcal{R}, (0, 0)) = (1, 0)$.

Case 3. $(\alpha, \beta) = (1, 1)$:

First we claim that $u + v \leq 2$ for any $(u, v) \in K$. We prove the claim by contradiction. Suppose there exists $(u, v) \in K$ such that $u + v > 2$. Then for any $0 \leq t \leq 1$, we have

$$t(u, v) + (1 - t)(1, 1) = ((u - 1)t + 1, (v - 1)t + 1) \in K$$

since K is convex. Let $g(t)$ be the value of g at the point $t(u, v) + (1 - t)(1, 1) \in K$ lying on the line segment joining $(1, 1)$ and (u, v).



Then

$$\begin{aligned} g(t) &= g(1 + (u-1)t, 1 + (v-1)t) \\ &= ((u-1)t + 1)((v-1)t + 1) \\ &= (u-1)(v-1)t^2 + (u+v-2)t + 1 \end{aligned}$$

We have

$$g'(t) = 2(u-1)(v-1)t + u + v - 2$$

which implies

$$g'(0) = u + v - 2 > 0$$

It follows that there exists $0 < t \leq 1$ such that

$$g(t) > g(0) = g(1, 1)$$

which contradicts that g attains its maximum at $(1, 1)$ on K. Hence we proved the claim that $u + v \leq 2$ for any $(u, v) \in K$. Now let $\mathcal{R}'$ be the convex hull of $\{(u, v) : (u, v) \in \mathcal{R} \text{ or } (v, u) \in \mathcal{R}\}$. Then $u' + v' \leq 2$ for any $(u', v') \in \mathcal{R}'$ since $u + v \leq 2$ for any $(u, v) \in \mathcal{R}$. By symmetry, we have $A'(\mathcal{R}', (0, 0)) = (\alpha', \alpha')$ for some $(\alpha', \alpha') \in \mathcal{R}'$. Now $\alpha' \leq 1$ since $\alpha' + \alpha' \leq 2$. Since $(1, 1) \in K \subset \mathcal{R}'$, we have $A'(\mathcal{R}', (0, 0)) = (1, 1)$ by Pareto optimality. Therefore $A'(\mathcal{R}, (0, 0)) = (1, 1)$ by independence of irrelevant alternative.

This completes the proof that $A'(\mathcal{R},(\mu,\nu)) = A(\mathcal{R},(\mu,\nu))$ for any closed and bounded convex set $\mathcal{R}$ and any point $(\mu,\nu) \in \mathcal{R}$. $\square$

Example 4.3.5 (Dating game)**.** *Consider the dating game given by the bimatrix*

$$(A,B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

We use $(\mu,\nu) = (\nu(A), \nu(B^T)) = (\frac{4}{5}, \frac{6}{5})$ as the status quo point (see Example 4.1.5). We need to find the payoff pair on

$$K = \left\{(u,v) \in \mathcal{R} : u \geq \frac{4}{5}, v \geq \frac{6}{5}\right\}$$

so that the function

$$g(u,v) = \left(u - \frac{4}{5}\right)\left(v - \frac{4}{5}\right)$$

attains its maximum. Now any payoff pair (u,v) along the line segment joining $(1,3)$ and $(4,2)$ satisfies

$$\begin{aligned} v - 3 &= -\frac{1}{3}(u-1) \\ v &= -\frac{1}{3}u + \frac{10}{3} \end{aligned}$$

Thus

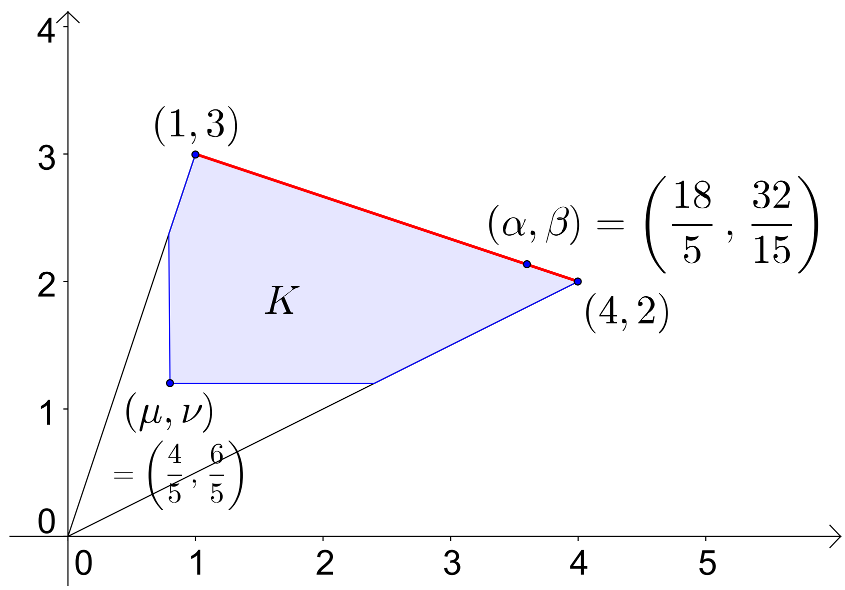
$$\begin{aligned} g(u,v) &= \left(u - \frac{4}{5}\right)\left(v - \frac{6}{5}\right) \\ &= \left(u - \frac{4}{5}\right)\left(-\frac{1}{3}u + \frac{32}{15}\right) \\ &= -\frac{1}{3}u^2 + \frac{12}{5}u - \frac{128}{75} \end{aligned}$$

attains its maximum when

$$u = \frac{18}{5} \text{ and } v = \frac{32}{15}$$

Since this payoff pair lies on the line segment joining $(1,3)$ and $(4,2)$, the arbitration pair of the game with status quo point $(\mu,\nu) = (\frac{4}{5}, \frac{6}{5})$ is

$$(\alpha,\beta) = \left(\frac{18}{5}, \frac{32}{15}\right)$$



□

To find the arbitration pair, one may use the fact that if $g(u, v) = (u - \mu)(v - \nu)$ attains it maximum at the point (α, β) over the line joining (u_0, v_0) and (u_1, v_1), then the slope of the line joining (α, β) and (μ, ν) would be equal to the negative of the slope of line joining (u_0, v_0) and (u_1, v_1). Using this fact, one may see easily that (α, β) satisfies

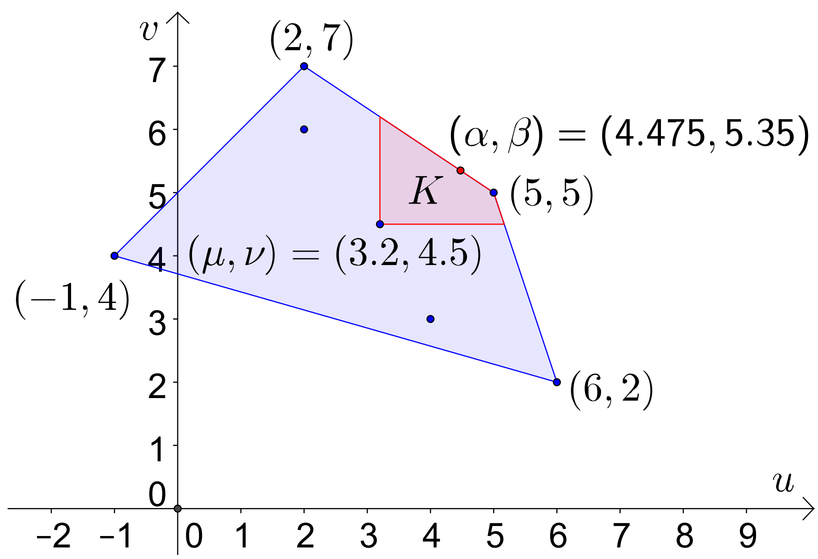
$$\begin{cases} \beta - v_0 = \dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - u_0) \\ \beta - \nu = -\dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - \mu) \end{cases}$$

Hence if the payoff pair (α, β) obtained by solving the above system of equations lies on the line segment joining (u_0, v_0) and (u_1, v_1), which implies that (α, β) lies on the bargaining set, then (α, β) is the arbitrary pair.

Example 4.3.6. *Let*

$$(A, B) = \begin{pmatrix} (2, 6) & (6, 2) & (-1, 4) \\ (4, 3) & (2, 7) & (5, 5) \end{pmatrix}$$

The reader may check that the values of A, B^T are 3.2, 4.5 respectively and we use $(\mu, \nu) = (3.2, 4.5)$ as the status quo point. We need to consider two line segments.



1. The line segment joining $(5, 5)$ and $(6, 2)$:

   The equation of the line segment is given by $v = -3u + 20$. The value of $g(u, v)$ along the line segment is

$$\begin{aligned} g(u,v) &= (u-3.2)(v-4.5) \\ &= (u-3.2)(-3u+15.5) \\ &= -3u^2 + 25.1u + 49.6 \end{aligned}$$

   which attain its maximum at $(\frac{251}{60}, \frac{149}{20})$. Since this payoff pair lies outside the line segment joining $(5, 5)$ and $(6, 2)$ and thus lies outside K, we know that the arbitration pair does not lie on the line segment joining $(5, 5)$ and $(6, 2)$.

2. The line segment joining $(2, 7)$ and $(5, 5)$:

   The slope of the line joining $(2, 7)$ and $(5, 5)$ is $-\frac{2}{3}$. To find the maximum point of $g(u, v)$ along the line joining $(2, 7)$ and $(5, 5)$, we may solve

$$\begin{cases} v - 7 = -\dfrac{2}{3}(u-2) \\ v - 4.5 = \dfrac{2}{3}(u-3.2) \end{cases}$$

which gives $(u, v) = (4.475, 5.35)$. Since this payoff pair lies on the line segment joining $(2, 7)$ and $(5, 5)$, we conclude that the arbitration pair is $(\alpha, \beta) = (4.475, 5.35)$.

$\square$

Exercise 4

1. Find all Nash equilibria of the following bimatrix games. For each of the Nash equilibrium, find the payoff pair.

   (a) $\begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$

   (b) $\begin{pmatrix} (5,2) & (2,0) \\ (1,1) & (3,4) \end{pmatrix}$

   (c) $\begin{pmatrix} (1,5) & (2,3) \\ (5,2) & (4,2) \end{pmatrix}$

2. The Brouwer's fixed-point theorem states that every continuous map $f : X \to X$ has a fixed-point if X is homeomorphic to a closed unit ball. Find a map $f : X \to X$ which does not have any fixed-point for each of the following topological spaces X. (It follows that the following spaces are not homeomorphic to a closed unit ball.)

   (a) X is the punched closed unit disc $D^2 \setminus \{0\} = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 1\}$

   (b) X is the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

   (c) X is the open unit disc $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

3. For each of the following bimatrices (A, B), find the values ν_A and ν_{B^T} of A and B^T respectively, and the Nash bargaining solution using $(\mu, \nu) = (\nu_A, \nu_{B^T})$ as the status quo point.

(a) $\begin{pmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{pmatrix}$ (c) $\begin{pmatrix} (2,2) & (0,1) & (1,-1) \\ (4,1) & (-2,1) & (1,3) \end{pmatrix}$

(b) $\begin{pmatrix} (3,1) & (1,0) \\ (0,-1) & (2,3) \end{pmatrix}$ (d) $\begin{pmatrix} (6,4) & (0,10) & (4,1) \\ (8,-2) & (4,1) & (0,1) \end{pmatrix}$

4. Two broadcasting companies, NTV and CTV, bid for the exclusive broadcasting rights of an annual sports event. If both companies bid, NTV will win the bidding with a profit of $20 (million) and CTV will have no profit. If only NTV bids, therell be a profit of $50 (million). If only CTV bids, therell be a profit of $40 (million). Find the Nash's solution to the bargaining problem.

5. Let $\mathcal{R} = \{(u,v) : v \geq 0 \text{ and } u^2 + v \leq 4\} \subset \mathbb{R}^2$. Find the arbitration pair $A(\mathcal{R}, (\mu, \nu))$ using the following points as the status quo point (μ, ν).

   (a) $(0,0)$ (b) $(0,1)$

6. Let $\mathcal{R} \subset \mathbb{R}^2$ be a closed and bounded convex set, $(\mu,\nu) \in \mathcal{R}$ and $(\alpha,\beta) = A(\mathcal{R}, (\mu,\nu))$ be the arbitration pair with $\alpha \neq \mu$. Suppose the boundary of $\mathcal{R}$ is given, locally at (α,β), by the graph of a differentiable function $f(x)$ with $f(\alpha) = \beta$. Prove that $f'(\alpha)$ is equal to the negative of the slope of the line joining (μ,ν) and (α,β).

5 Cooperative games

In a **cooperative game**, players can make binding agreements about which strategies to play. In the last chapter, we studied Nash bargaining solution for 2-person cooperative games with non-transferable utility. In this chapter, we study n-person cooperative games with **transferable utility**. In such a game, players may share their payoffs according to the agreements made by the players in advance. However there is no universally accepted rules to determine how the payoffs should be shared among the players. Different solution concepts may be used in different situations. In this chapter, we are going to study two solution concepts namely core and Shapley value.

5.1 Characteristic form and imputations

First we define the strategic form of a cooperative game.

Definition 5.1.1. *Let $A = \{A_1, A_2, \cdots, A_n\}$ be the set of players. Let X_i, $i = 1, 2, \cdots, n$, be the set of strategies of player $A_i \in A$.*

1. *The* **strategic form** *of a game is a function*

$$\pi = (\pi_1, \pi_2, \cdots, \pi_n) : X_1 \times X_2 \times \cdots \times X_n \to \mathbb{R}^n$$

2. *A* **coalition** *is a subset $S \subset A$ of the set of players. For each $i = 1, 2, \cdots, n$, the set $\{A_i\}$, consists of one player, is a coalition. The whole set A of all players is also a coalition which is called the* **grand coalition**.

3. *Let $S \subset A$ be a coalition. The* **counter coalition** *of S is the complement $S^c = A \setminus S \subset A$ of S in A.*

4. *The* **characteristic function** *is the function $\nu : \mathcal{P}(A) \to \mathbb{R}$, where $\mathcal{P}(A)$ is the power set of A, defined as follows. For any coalition $S \subset A$, define $\nu(S)$ as the maximin total payoff to the players in S when the game is considered as a 2-person non-cooperative game between S and S^c. For a coalition with one single player $S = \{A_i\}$, $A_i \in A$, we will use an abuse of notation and write $\nu(A_i)$ for $\nu(\{A_i\})$.*

Example 5.1.2 (3-person constant sum game)**.** *Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.*

Strategy	*Payoff vector*
$(0,0,0)$	$(-2,1,2)$
$(0,0,1)$	$(1,1,-1)$
$(0,1,0)$	$(0,-1,2)$
$(0,1,1)$	$(-1,2,0)$
$(1,0,0)$	$(1,-1,1)$
$(1,0,1)$	$(0,0,1)$
$(1,1,0)$	$(1,0,0)$
$(1,1,1)$	$(1,2,-2)$

For coalition $S = \{A_1, A_2\}$, we compute $\nu(S)$ and $\nu(S^c)$ as follows. First the game bimatrix for the 2-person game between S and S^c is

		Strategy of A_3	
		0	1
Strategy of $\{A_1, A_2\}$	$(0,0)$	$(-1,2)$	$(2,-1)$
	$(0,1)$	$(-1,2)$	$(1,0)$
	$(1,0)$	$(0,1)$	$(0,1)$
	$(1,1)$	$(1,0)$	$(3,-2)$

The game has a saddle point with payoff pair $(1,0)$. Thus $\nu(\{A_1, A_2\}) = 1$ and $\nu(\{A_3\}) = 0$. For $S = \{A_1, A_3\}$, the game bimatrix is

		Strategy of A_2	
		0	1
Strategy of $\{A_1, A_3\}$	$(0,0)$	$(0,1)$	$(2,-1)$
	$(0,1)$	$(0,1)$	$(-1,2)$
	$(1,0)$	$(2,-1)$	$(1,0)$
	$(1,1)$	$(1,0)$	$(-1,2)$

Now the payoff matrix for the coalition $S = \{A_1, A_3\}$ is

$$\begin{pmatrix} 0 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Observe that the sum of the payoffs to S and S^c is always equal to 1. The non-cooperative game between S and S^c can be considered as a zero sum game. The value of $\nu(\{A_2, A_3\})$ is equal to the value of the above game matrix which is equal to $\frac{4}{3}$. Moreover, $\nu(\{A_1\}) = -\frac{1}{3}$ since the sum of the payoffs to S and S^c is always equal to 1. The values of $\nu(S)$ for varies coalitions S are given in the following table.

S	$\nu(S)$
$\emptyset$	0
$\{A_1\}$	$\frac{1}{4}$
$\{A_2\}$	$-\frac{1}{3}$
$\{A_3\}$	0
$\{A_1, A_2\}$	1
$\{A_2, A_3\}$	$\frac{3}{4}$
$\{A_1, A_3\}$	$\frac{4}{3}$
$\{A_1, A_2, A_3\}$	1

□

Suppose S and T are two disjoint coalitions. The two coalitions can combine and form a larger coalition $S \cup T$ which is called the **union coalition**. We always have $\nu(S \cup T) \geq \nu(S) + \nu(T)$. This property is called superadditivity.

Theorem 5.1.3 (Superadditivity)**.** *Let ν be the characteristic function of a game in strategic form. Then ν is* **superadditive**. *That is to say, if $S, T \subset A$ are two coalitions with $S \cap T = \emptyset$, then*

$$\nu(S \cup T) \geq \nu(S) + \nu(T)$$

In particular

$$\nu(A) \geq \sum_{i=1}^{n} \nu(A_i)$$

Proof. Let S and T be two coalitions with $S \cap T = \emptyset$. Let $\mathbf{p}$ and $\mathbf{q}$ be the maximin strategies for the coalitions S and T respectively. By combining $\mathbf{p}$ and $\mathbf{q}$ which is a strategy of $S \cup T$, the coalition $S \cup T$ may guarantee a payoff of at least $\nu(S) + \nu(T)$. Therefore we have $\nu(S \cup T) \geq \nu(S) + \nu(T)$. The second statement is a direct consequence of the first. □

Definition 5.1.4 (Characteristic form)**.** *The* **characteristic form** *of a game is an ordered pair (A, ν) where A is the set of player and $\nu : \mathcal{P}(A) \to \mathbb{R}$, where $\mathcal{P}(A)$ is the power set of A, is a function, such that*

1. $\nu(\emptyset) = 0$
2. *(Superadditivity) If $S, T \subset A$ are subset of A with $S \cap T = \emptyset$, then*

$$\nu(S \cup T) \geq \nu(S) + \nu(T)$$

The function ν is called the **characteristic function** *of the game.*

The players have a tendency to cooperate only when the game is essential.

Definition 5.1.5. *We say that a game (A, ν) in characteristic form is* **essential** *if*

$$\nu(A) > \sum_{i=1}^{n} \nu(A_i)$$

Otherwise, it is said to be **inessential**.

If a game is essential, then the total payoff to all players when they cooperate is larger than the sum of the payoffs to the players when they play the game individually. This gives an incentive for the players to cooperate. If a game is inessential, then no player can gain more by cooperation.

Theorem 5.1.6. *If (A, ν) is inessential, then for any coalition $S \subset A$, we have*

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

Proof. For any coalition $S \subset A$, by superadditivity, we have

$$\nu(S) \geq \sum_{A_i \in S} \nu(A_i) \text{ and } \nu(S^c) \geq \sum_{A_j \in S^c} \nu(A_j)$$

Now if (A, ν) is inessential, then

$$\nu(A) \leq \sum_{i=1}^{n} \nu(A_i)$$

which implies, by superadditivity again,

$$\nu(A) = \sum_{i=1}^{n} \nu(A_i)$$

Hence

$$\begin{aligned}\nu(A) &= \sum_{i=1}^{n} \nu(A_i) \\ &= \sum_{A_i \in S} \nu(A_i) + \sum_{A_j \in S^c} \nu(A_j) \\ &\leq \nu(S) + \nu(S^c) \\ &\leq \nu(A)\end{aligned}$$

Thus all inequalities above become equality and therefore

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

□

In a cooperative game with transferable utility, the players may benefit by forming the grand coalition A. The total amount received by the players is $\nu(A)$. The problem is to agree on how this amount should be split among the players. The first criterion is that each player should receive no less than the amount before cooperation. We call a splitting of total payoffs to the players an imputation if it satisfies this criterion.

Definition 5.1.7 (Imputation). *Let $\nu : \mathcal{P}(A) \to \mathbb{R}$ be a characteristic function. A vector $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is called an* **imputation** *for ν if*

1. *(Individual rationality) For any $i = 1, 2, \cdots, n$, we have $x_i \geq \nu(A_i)$.*
2. *(Efficiency[4]) $\sum_{i=1}^{n} x_i = \nu(A)$*

The set of imputations for ν is denoted by $I(\nu)$.

In an inessential game, no player may receive more by cooperation and there is only one imputation for the game. For essential games, there are infinitely many ways to split the payoffs which satisfy individual rationality.

Theorem 5.1.8. *Let ν be a characteristic function and $I(\nu)$ be the set of imputations.*

[4]It is also called collective rationality.

1. *If ν is inessential, then $I(\nu) = \{(\nu(A_1), \nu(A_2), \cdots, \nu(A_n))\}$.*

2. *If ν is essential, then $I(\nu)$ is an infinite set.*

Proof. 1. If ν is inessential, then for any imputation $(x_1, x_2, \cdots, x_n) \in I(\nu)$, we have

$$\nu(A) = \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} \nu(A_i) = \nu(A)$$

Thus $x_i = \nu(A_i)$ for $i = 1, 2, \cdots, n$ and $I(\nu) = \{(\nu(A_1), \nu(A_2), \cdots, \nu(A_n))\}$.

2. Suppose ν is essential. Let

$$\beta = \nu(A) - \sum_{i=1}^{n} \nu(A_i) > 0$$

Then there are infinitely many solutions to the equation $\sum_{i=1}^{n} \alpha_i = \beta$ for variables $\alpha_1, \alpha_2, \cdots, \alpha_n > 0$ and each of the solutions gives an imputation by putting $x_i = \nu(A_i) + \alpha_i$ for $i = 1, 2, \cdots, n$.

□

5.2 Core

The core of a cooperative game is the set of imputations that are not dominated by other imputations through any coalition.

Definition 5.2.1. *Let $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in I(\nu)$ be two imputations. We say that $\mathbf{x}$ is dominated by $\mathbf{y}$ through a coalition $S \subset A$ and write $\mathbf{x} \prec_S \mathbf{y}$ if*

1. *If $A_i \in S$, then $x_i < y_i$.*

2. $\sum_{A_i \in S} y_i \leq \nu(S)$

We write $\mathbf{x} \not\prec_S \mathbf{y}$ if $\mathbf{x}$ is not dominated by $\mathbf{y}$ through S.

Example 5.2.2. *Consider the 3-person constant sum game (Example 5.1.2) with characteristic function*

S	$\nu(S)$
$\emptyset$	0
$\{A_1\}$	$\frac{1}{4}$
$\{A_2\}$	$-\frac{1}{3}$
$\{A_3\}$	0
$\{A_1, A_2\}$	1
$\{A_2, A_3\}$	$\frac{3}{4}$
$\{A_1, A_3\}$	$\frac{4}{3}$
$\{A_1, A_2, A_3\}$	1

We have

$$\begin{aligned}
\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) &\prec_{\{A_1, A_2\}} \left(\frac{1}{2}, \frac{1}{2}, 0\right) \\
(1, 0, 0) &\prec_{\{A_2, A_3\}} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) &\prec_{\{A_2, A_3\}} \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)
\end{aligned}$$

□

For imputation $\mathbf{x} \in I(\nu)$, if there exists imputation $\mathbf{y} \in I(\nu)$ and coalition $S \subset A$ such that $\mathbf{x} \not\prec_S \mathbf{y}$, then there will be a tendency for coalition S to form and upset the proposal $\mathbf{x}$ because such a coalition could guarantee each of its members more than they could receive from $\mathbf{x}$. Thus it reasonable to require the splitting of payoff to the players to be an imputation which is not dominated by any other imputation through any coalition.

Definition 5.2.3 (Core)**.** *The* **core** $C(\nu)$ *of a characteristic function is the set of all imputations that are not dominated by any other imputation through any coalition, that is*

$$C(\nu) = \{\mathbf{x} \in I(\nu) : \mathbf{x} \not\prec_S \mathbf{y} \textit{ for any } \mathbf{y} \in I(\nu) \textit{ and } S \subset A\}$$

There is an easy way to check whether an imputation lies in the core.

Theorem 5.2.4. *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ *be an imputation. Then* $\mathbf{x} \in C(\nu)$ *if and only if*

$$\sum_{A_i \in S} x_i \geq \nu(S)$$

for any coalition $S \subset A$*.*

Proof. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ does not lie in the core $C(\nu)$. Then there exists imputation $\mathbf{y} \in I(\nu)$ and coalition $S \subset A$ such that $x_i < y_i$ for any $A_i \in S$ and $\sum_{A_i \in S} y_i \leq \nu(S)$. Thus we have

$$\sum_{A_i \in S} x_i < \sum_{A_i \in S} y_i \leq \nu(S)$$

On the other hand, suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ is an imputation such that

$$\sum_{A_i \in S} x_i < \nu(S)$$

for some coalition S. Then $S \neq A$ and since

$$\sum_{A_i \in S} x_i + \sum_{A_j \in S^c} x_j = \sum_{i=1}^{n} x_i = \nu(A) \geq \nu(S) + \nu(S^c) > \sum_{A_i \in S} x_i + \sum_{A_j \in S^c} \nu(A_j)$$

by superadditivity, there exists $A_k \in S^c$ such that $x_k > \nu(A_k)$. Define

$$y_i = \begin{cases} x_i + \frac{\alpha}{|S|} & \text{for } A_i \in S \\ x_k - \alpha & \text{for } i = k \\ x_i & \text{for } A_i \in S^c \text{ and } i \neq k \end{cases}$$

where

$$\alpha = \min \left\{ x_k - \nu(A_k), \nu(S) - \sum_{A_i \in S} x_i \right\} > 0$$

By taking $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, we have $\mathbf{x} \prec_S \mathbf{y}$. Therefore $\mathbf{x}$ does not lie in the core $C(\nu)$ and the proof of the theorem is complete. □

Theorem 5.2.5. *The core $C(\nu)$ is a convex set if it is not empty.*

Proof. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in C(\nu)$ be two imputations in the core. Then for any coalition S, we have

$$\sum_{A_i \in S} x_i, \sum_{A_i \in S} y_i \geq \nu(S)$$

by Theorem 5.2.4. Now for any real number $0 \leq \lambda \leq 1$, we have

$$\sum_{A_i \in S} (\lambda x_i + (1 - \lambda) y_i) \geq \nu(S)$$

which implies $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C(\nu)$. Therefore $C(\nu)$ is convex. □

Example 5.2.6 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.2.2). Find the core $C(\nu)$ of ν.*

Solution. For any imputation $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$, we have $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq \dfrac{1}{4}, x_2 \geq -\dfrac{1}{3}, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_2 + x_3 \geq \dfrac{3}{4}, x_1 + x_3 \geq \dfrac{4}{3} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

First of all, we have

$$x_3 = (x_1 + x_2 + x_3) - (x_1 + x_2) \leq 1 - 1 = 0$$

which implies $x_3 = 0$. Then

$$x_1 + x_2 = (x_1 + x_3) + (x_2 + x_3) \geq \frac{4}{3} + \frac{3}{4} > 1$$

which leads to a contradiction. Therefore $C(\nu) = \emptyset$. □

Example 5.2.7. *Suppose $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$ and*

S	$\nu(S)$
$\{A_1, A_2\}$	$\frac{1}{3}$
$\{A_1, A_3\}$	$\frac{1}{2}$
$\{A_2, A_3\}$	$\frac{1}{4}$
$\{A_1, A_2, A_3\}$	1

Find the core $C(\nu)$ of ν.

Solution. Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation. Then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq \dfrac{1}{3}, x_1 + x_3 \geq \dfrac{1}{2}, x_2 + x_3 \geq \dfrac{1}{4} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

Now

$$\begin{array}{l} 0 \leq x_1 = 1 - x_2 - x_3 \leq 1 - \frac{1}{4} = \frac{3}{4} \\ 0 \leq x_2 = 1 - x_1 - x_3 \leq 1 - \frac{1}{2} = \frac{1}{2} \\ 0 \leq x_3 = 1 - x_1 - x_2 \leq 1 - \frac{1}{3} = \frac{2}{3} \end{array}$$

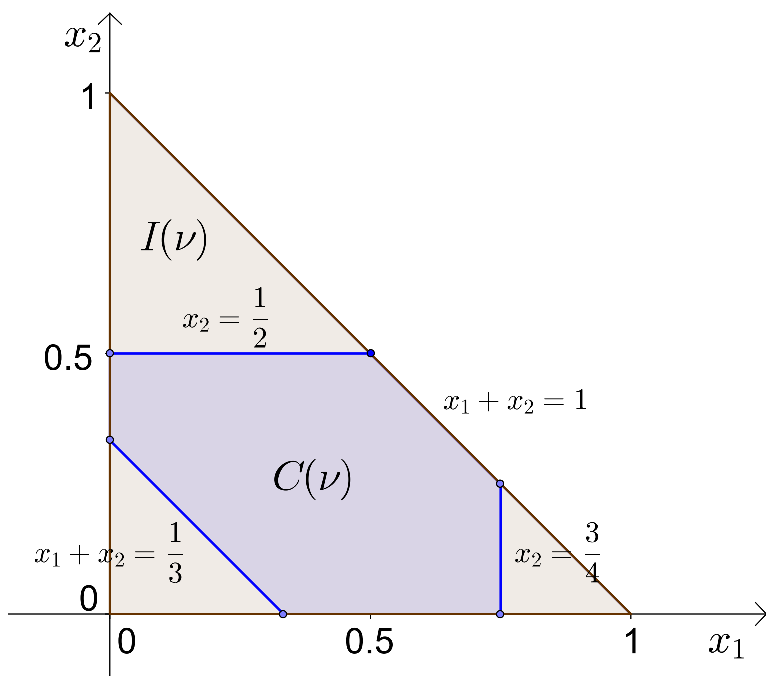
The above system of inequalities is equivalent to

$$\begin{cases} 0 \le x_1 \le \frac{3}{4} \\ 0 \le x_2 \le \frac{1}{2} \\ 0 \le x_3 \le \frac{2}{3} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

We may consider x_1 and x_2 as independent variables and $x_3 = 1 - x_1 - x_2$ depends on x_1, x_2. Then x_1 and x_2 satisfy the constraints

$$\begin{cases} 0 \le x_1 \le \frac{3}{4} \\ 0 \le x_2 \le \frac{1}{2} \\ \dfrac{1}{3} \le x_1 + x_2 \le 1 \end{cases}$$

We may represent the core on the $x_1 - x_2$ plane



□

Example 5.2.8 (Used car game). *A man named Andy has an old car he wishes to sell. He no longer drives it, and it is worth nothing to him unless he can sell it. Two people are interested in buying it, Ben and Carl. Bill values the car at \$500 and Carl thinks it is worth \$700. The game consists of each of the prospective buyers bidding on the car, and Andy either accepting one of the bids (presumably the higher one), or rejecting both of them. Find the core of the game and represent it on the* $x_1 - x_2$ *plane.*

Solution. If there is no deal, no player gets anything and we have $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$. The characteristic values of other coalitions are listed below.

S	$\nu(S)$
$\{A_1, A_2\}$	500
$\{A_1, A_3\}$	700
$\{A_2, A_3\}$	0
$\{A_1, A_2, A_3\}$	700

Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation. Then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 500, x_1 + x_3 \geq 700, x_2 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = 700 \end{cases}$$

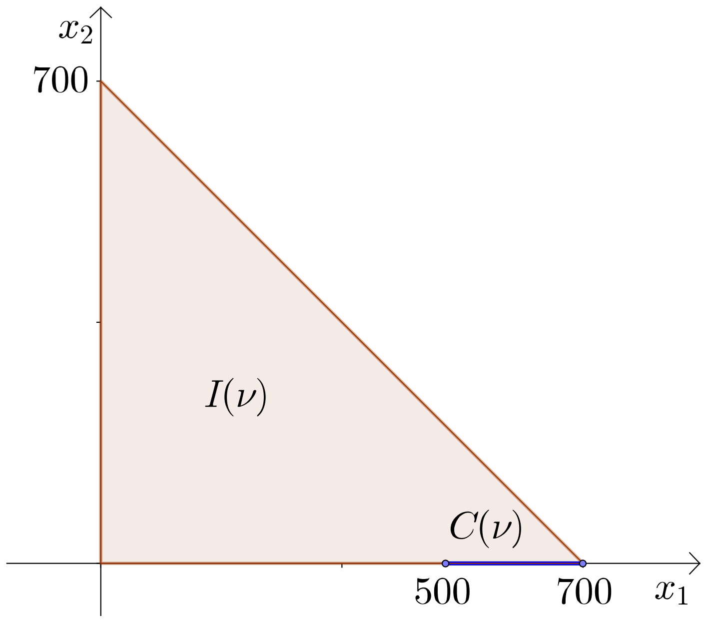
Observe that

$$\begin{array}{l} 0 \leq x_2 = (x_1 + x_2 + x_3) - (x_1 + x_3) \leq 700 - 700 = 0 \\ 0 \leq x_3 = (x_1 + x_2 + x_3) - (x_1 + x_2) \leq 700 - 500 = 200 \\ x_1 = x_1 + x_2 \geq 500 \\ x_1 \leq x_1 + x_2 + x_3 \leq 700 \end{array}$$

The above system of inequalities is equivalent to

$$\begin{cases} 500 \leq x_1 \leq 700 \\ x_2 = 0 \\ x_3 = 700 - x_1 \end{cases}$$

The core of the used car game is shown in the following figure.



□

Example 5.2.9 (Mayor and council)**.** *In a city, there is a Mayor and a city council with 7 members. A bill can be passed to a law if either*

1. *the majority of the council members passes it and the Mayor signs it, or*

2. *the Mayor vetoes it but at least 6 council members vote to override the veto.*

Find the core of the game.

Solution. Let $A = \{M, 1, 2, 3, 4, 5, 6, 7\}$ be the set of players. Then

- $\nu(S) = 1$ if
  1. S contains the mayor and at least 4 council members, or
  2. S contains at least 6 council members.
- $\nu(S) = 0$ otherwise.

Then $\mathbf{x} = (x_M, x_1, \cdots, x_7) \in I(\nu)$ if and only if

$$\begin{cases} x_M, x_1, x_2, \cdots, x_7 \geq 0 \\ x_M + x_1 + x_2 + \cdots + x_7 = 1 \end{cases}$$

Suppose $\mathbf{x} \in C(\nu)$. Then for any $k = 1, 2, \cdots, 7$,

$$\sum_{i \neq k} x_i \geq 1$$

which implies $x_1 + x_2 + \cdots + x_7 \geq 1$ and

$$x_M = x_M + x_1 + x_2 \cdots + x_7 - (x_1 + x_2 + \cdots + x_7) \leq 1 - 1 = 0$$

Moreover for any $k = 1, 2, \cdots, 7$,

$$x_k = (x_1 + x_2 + \cdots + x_7) - \sum_{i \neq k} x_i \leq 1 - 1 = 0$$

which contradicts $x_M + x_1 + x_2 + \cdots + x_7 = 1$. Therefore $C(\nu) = \emptyset$. □

Definition 5.2.10. *A characteristic function ν is* **constant sum** *if*

$$\nu(S) + \nu(S^c) = \nu(A)$$

for any coalition $S \subset A$.

Theorem 5.2.11. *If ν is both essential and constant sum, then $C(\nu) = \emptyset$.*

Proof. Suppose ν is constant sum and its core $C(\nu)$ is nonempty. It suffices to show that ν is inessential. To this end, let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in C(\nu)$ be an imputation lying in the core. Then for any $k = 1, 2, \cdots, n$, we have

$$x_k \geq \nu(A_k) \text{ and } \sum_{i \neq k} x_i \geq \nu(\{A_k\}^c)$$

Thus by Theorem 5.2.4, we have

$$\nu(A) = \nu(A_k) + \nu(\{A_k\}^c) \leq x_k + \sum_{i \neq k} x_i = \nu(A)$$

It follows that $x_k = \nu(A_k)$ and we have

$$\sum_{k=1}^{n} \nu(A_k) = \nu(A)$$

which means ν is inessential and the proof of the theorem is complete. □

Definition 5.2.12. *Two characteristic functions μ and ν are* **strategically equivalent** *if there exists real numbers $k > 0$ and $c_1, c_2, \cdots, c_n$ such that for any coalition $S \subset A$,*

$$\mu(S) = k\nu(S) + \sum_{A_i \in S} c_i$$

It is obvious that being strategically equivalent is an equivalence relation. Two games share very similar properties when their characteristic functions are strategically equivalent.

Theorem 5.2.13. *Suppose μ and ν are strategically equivalent characteristic functions. Let $k > 0$, $c_1, c_2, \cdots, c_n$ be real numbers such that*

$$\mu(S) = k\nu(S) + \sum_{A_i \in S} c_i$$

Write $\mathbf{c} = (c_1, c_2, \cdots, c_n)$. We have

1. *μ is essential if and only if ν is essential.*
2. *$I(\mu) = \{\mathbf{y} : \mathbf{y} = k\mathbf{x} + \mathbf{c}$ for some $\mathbf{x} \in I(\nu)\}$*
3. *$C(\mu) = \{\mathbf{y} : \mathbf{y} = k\mathbf{x} + \mathbf{c}$ for some $\mathbf{x} \in C(\nu)\}$*

Definition 5.2.14 ($(0,1)$ reduced form)**.** *We say that a characteristic function μ is a $(0,1)$* **reduced form** *if*

1. *$\mu(A_i) = 0$ for any $i = 1, 2, \cdots, n$*
2. *$\mu(A) = 1$*

Every inessential game is strategically equivalent to a trivial game. Every essential game is strategically equivalent to a unique $(0,1)$ reduced form.

Theorem 5.2.15. *Let ν be a characteristic function.*

1. *If ν is inessential, then ν is strategically equivalent to the zero game, that is, a game with characteristic function identically equal to zero.*
2. *If ν is essential, then ν is strategically equivalent to a unique game in $(0,1)$ reduced form.*

Proof. Let ν be a characteristic function.

1. Suppose ν is inessential. By Theorem 5.1.6, for any coalition $S \subset A$,

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

Taking $k = 1$ and $c_i = -\nu(A_i)$ for $i = 1, 2, \cdots, n$, we have ν is strategically equivalent to the characteristic function

$$\mu(S) = \nu(S) - \sum_{A_i \in S} \nu(A_i)$$

and $\mu(S) = 0$ for any coalition S which means μ is the trivial game.

2. Suppose ν is essential. Taking

$$k = \frac{1}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)} \text{ and } c_i = \frac{-\nu(A_i)}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)} \text{ for } i = 1, 2, \cdots, n$$

ν is strategically equivalent to the characteristic function

$$\mu(S) = \frac{\nu(S) - \sum_{A_i \in S} \nu(A_i)}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)}$$

for $S \subset A$. Now $\mu(A) = 1$ and $\mu(A_i) = 0$ for any $i = 1, 2, \cdots, n$. Therefore ν is strategically equivalent to the $(0, 1)$ reduced form μ. Suppose μ' is another $(0, 1)$ reduced form strategically equivalent to ν. Then μ' is strategically equivalent to μ. Thus there exists constants $k > 0$ and $c_1, c_2, \cdots, c_n$ such that

$$\mu'(S) = k\mu(S) + \sum_{A_i \in S} c_i$$

for any coalition S. Taking $S = \{A_i\}$, $i = 1, 2, \cdots, n$, we have $c_1 = c_2 = \cdots = c_n = 0$ since $\mu'(\{A_i\}) = \mu(\{A_i\}) = 0$. Moreover taking $S = A$, we have $\mu'(A) = k\mu(A)$ which implies $k = 1$ since $\mu'(A) = \mu(A) = 1$. Therefore $\mu' = \mu$ and the $(0, 1)$ reduced form of ν is unique.

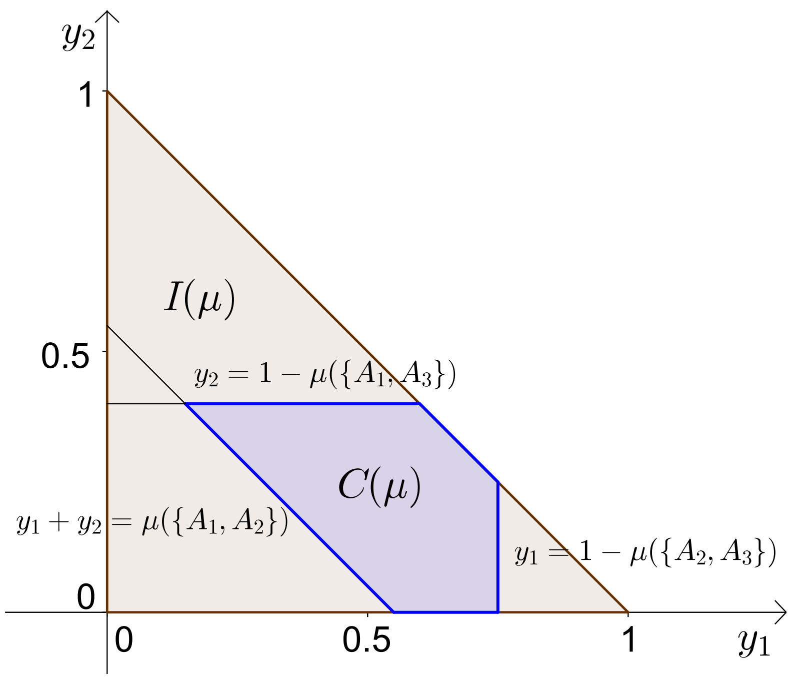
□

Suppose ν is the characteristic function of a 3-person game and μ is the $(0,1)$ reduced form of ν. Then an imputation $(y_1, y_2, y_3) \in I(\mu)$ of μ lies in the core of μ if and only if

$$\begin{cases} 0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) \\ 0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) \\ 0 \le y_3 \le 1 - \mu(\{A_1, A_2\}) \\ y_1 + y_2 + y_3 = 1 \end{cases}$$

and on the $x_1 - x_2$ plane, it can be represented by the region

$$\begin{cases} 0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) \\ 0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) \\ \mu(\{A_1, A_2\}) \le y_1 + y_2 \le 1 \end{cases}$$



Example 5.2.16 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.1.2 and Example 5.2.2). Let μ be the $(0,1)$ reduced form of ν. Find μ and its core $C(\mu)$.*

Solution. First we have

$$\mu(A_1) = \mu(A_2) = \mu(A_3) = 0$$

and

$$\mu(A) = 1$$

Next we calculate

$$k = \frac{1}{\nu(A) - (\nu(A_1) + \nu(A_2) + \nu(A_3))} = \frac{1}{1 - (\frac{1}{4} + (-\frac{1}{3}) + 0)} = \frac{12}{13}$$

and we have

$$\begin{aligned}
\mu(\{A_1, A_2\}) &= k(\nu(\{A_1, A_2\}) - (\nu(A_1) + \nu(A_2))) \\
&= \frac{12}{13}\left(1 - \left(\frac{1}{4} - \frac{1}{3}\right)\right) \\
&= 1 \\
\mu(\{A_1, A_3\}) &= k(\nu(\{A_1, A_3\}) - (\nu(A_1) + \nu(A_3))) \\
&= \frac{12}{13}\left(\frac{4}{3} - \left(\frac{1}{4} + 0\right)\right) \\
&= 1 \\
\mu(\{A_2, A_3\}) &= k(\nu(\{A_2, A_3\}) - (\nu(A_2) + \nu(A_3))) \\
&= \frac{12}{13}\left(\frac{3}{4} - \left(-\frac{1}{3} + 0\right)\right) \\
&= 1
\end{aligned}$$

Now an imputation $(y_1, y_2, y_3) \in I(\mu)$ lies in the core $C(\mu)$ of μ if and only if

$$\begin{cases} 0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) = 0 \\ 0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) = 0 \\ 0 \le y_3 \le 1 - \mu(\{A_1, A_2\}) = 0 \\ y_1 + y_2 + y_3 = 1 \end{cases}$$

which has no solution. Thus $C(\mu) = \emptyset$. (Note that $C(\nu)$ is also empty.) □

Example 5.2.17 (Used car game)**.** *Let ν be the characteristic function of the used car game (Example 5.2.8). Let μ be the* $(0, 1)$ *reduced form of ν. Find μ and the core $C(\nu)$ of ν.*

Solution. First we have

$$\mu(A_1) = \mu(A_2) = \mu(A_3) = 0 \text{ and } \mu(A) = 1$$

Now

$$k = \frac{1}{\nu(A) - (\nu(A_1) + \nu(A_2) + \nu(A_3))} = \frac{1}{700}$$

and we have

$$\begin{aligned}
\mu(\{A_1, A_2\}) &= k(\nu(\{A_1, A_2\}) - (\nu(A_1) + \nu(A_2))) \\
&= \frac{500 - 0}{700} \\
&= \frac{5}{7} \\
\mu(\{A_1, A_3\}) &= k(\nu(\{A_1, A_3\}) - (\nu(A_1) + \nu(A_3))) \\
&= \frac{700 - 0}{700} \\
&= 1 \\
\mu(\{A_2, A_3\}) &= k(\nu(\{A_2, A_3\}) - (\nu(A_2) + \nu(A_3))) \\
&= \frac{0 - 0}{700} \\
&= 0
\end{aligned}$$

Now an imputation $(y_1, y_2, y_3) \in I(\mu)$ lies in the core $C(\mu)$ of μ if and only if

$$\begin{cases}
0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) = 1 - 0 = 1 \\
0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) = 1 - 1 = 0 \\
0 \le y_3 \le 1 - \mu(\{A_1, A_2\}) = 1 - \frac{5}{7} = \frac{2}{7} \\
y_1 + y_2 + y_3 = 1
\end{cases}$$

which is equivalent to

$$\begin{cases}
\frac{5}{7} \le y_1 \le 1 \\
y_2 = 0 \\
y_3 = 1 - y_1
\end{cases}$$

□

5.3 Shapley value

In the last section, we studied cores of characteristic functions. The core has a disadvantage that it may be empty and usually contains an infinite number of elements when it is nonempty. In this section, we study another solution concept called Shapley value which always exists and is unique.

Definition 5.3.1 (Shapley value)**.** *Let ν be a characteristic function. The* **Shapley value** *of the player A_k, $k = 1, 2, \cdots, n$, is defined as*

$$\phi_k = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} \frac{(n - |S|)!(|S| - 1)!}{n!} (\nu(S) - \nu(S \setminus \{A_k\}))$$

The vector $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ is called the **Shapley vector** *of ν.*

The Shapley value of a player can be interpreted in the following way. Suppose we form the grand coalition A by entering the players one after another. As player A_k enters the coalition, he receives the amount by which his entry inceases the value of the coalition he enters. This amount is equal to $\delta_k(S) = \nu(S) - \nu(S \setminus \{A_k\})$ where S is the coalition after A_k has entered. The amount a player receives depends on the order in which the players are entered. The Shapley value ϕ_k is the average amount that A_k receives over all orders of entering of players in forming the grand coalition.

Let S be a coalition which contains player A_k. There are $(|S|-1)!$ number of ways for other players in S to enter the coalition before A_k. Then player A_k enters the coalition to form the coalition S and there are $(n-|S|)!$ number of ways for the remaining players to enter to form the grand coalition. Thus among all $n!$ permutations of players in forming the grand coalition, there are $(n - |S|)!(|S| - 1)!$ of which the coalition S would form at the moment that player A_k enters into the coalition and A_k would receive $\nu(S) - \nu(S \setminus \{A_k\})$. Therefore the average amount that A_k receives is given by the formula in Definition 5.3.1. This also shows the following alternative formula for the Shapley values.

Theorem 5.3.2. *The Shapley value of the player A_k is given by*

$$\phi_k = \frac{1}{n!} \sum_{\sigma \in S_n} (\nu(S_k^\sigma) - \nu(S_k^\sigma \setminus \{A_k\}))$$

where S_n is the set of all permutations of $1, 2, \cdots, n$, and $S_k^\sigma = \{A_{\sigma(1)}, A_{\sigma(2)}, \cdots, A_{\sigma(i)}\}$ where i is determined by $\sigma(i) = k$. In other words, S_k^σ is the set of players in A which precede A_k in permutation σ, including A_k.

Remarks:

1. The quantity $\delta_k(S) = \nu(S) - \nu(S \setminus \{A_k\})$ is the amount the player A_k contributes to the coalition S. In particular $\delta_k(S) = 0$ if $A_k \notin S$. Therefore to find ϕ_k, we only need to sum over S with $A_k \in S$.

2. The formula for ϕ_k can also be written as

$$\phi_k = \sum_{S \in \mathcal{P}(A) \setminus \{A\}} \frac{(n - |S| - 1)!|S|!}{n!} (\nu(S \cup \{A_k\}) - \nu(S))$$

3. Suppose $n = 3$ and $\nu(A_k) = 0$ for $k = 1, 2, 3$. To find the Shapley value ϕ_1 of A_1, we need to calculate, for each permutation of players, the value of $\delta_1(S)$ where S is the coalition right after the joining of A_1. The values of $\delta_1(S)$ for the permutations of players are shown in the following table.

Permutation	S	$S \setminus \{A_1\}$	$\delta_1(S)$
123	$\{A_1\}$	$\emptyset$	0
132	$\{A_1\}$	$\emptyset$	0
213	$\{A_1, A_2\}$	$\{A_2\}$	$\nu(\{A_1, A_2\})$
231	$\{A_1, A_2, A_3\}$	$\{A_2, A_3\}$	$\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})$
312	$\{A_1, A_3\}$	$\{A_3\}$	$\nu(\{A_1, A_3\})$
321	$\{A_1, A_2, A_3\}$	$\{A_2, A_3\}$	$\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})$

The Shapley value ϕ_1 of A_1 is the average value in the last column. Thus we have

$$\phi_1 = \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_1, A_3\}) - 2\nu(\{A_2, A_3\})}{6}$$

We have similar formula for ϕ_2 and ϕ_3.

Now we prove that the Shapley vector is always an imputation.

Theorem 5.3.3. *Let ν be a characteristic function and $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ be the Shapley vector of ν. Then $\phi \in I(\nu)$. In other words, we always have*

1. $\phi_i \geq \nu(A_i)$ *for any* $i = 1, 2, \cdots, n$

2. $\sum_{i=1}^{n} \phi_i = \nu(A)$

Proof. 1. For any A_i and any coalition $S \subset A$ with $A_i \in S$, we have

$$\nu(A_i) + \nu(S \setminus \{A_i\}) \leq \nu(S)$$

by supperadditivity. Therefore

$$\begin{aligned}
\phi_i &= \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\
&\geq \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\nu(A_i) \\
&= \nu(A_i)
\end{aligned}$$

2.

$$\begin{aligned}
\sum_{i=1}^{n} \phi_i &= \sum_{i=1}^{n} \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(n-|S|)!(|S|-1)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\
&= \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\
&= \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\
&\quad - \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S \setminus \{A_i\}) \\
&= \sum_{S \subset A} |S| \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\
&\quad - \sum_{T \subset A} \sum_{A_j \notin T} \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\
&= \sum_{S \subset A} |S| \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\
&\quad - \sum_{T \subset A} (n-|T|) \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\
&= \sum_{S \subset A} \frac{(n-|S|)!|S|!}{n!}\nu(S) \\
&\quad - \sum_{\substack{T \subset A \\ T \neq A}} \frac{(n-|T|)!|T|!}{n!}\nu(T) \\
&= \nu(A)
\end{aligned}$$

□

Example 5.3.4 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.2.2). Find the Shapley value of each player.*

Solution. To find the Shapley value ϕ_1 of A_1, observe that the coalitions containing A_1 are $\{A_1\}$, $\{A_1, A_2\}$, $\{A_1, A_3\}$ and $\{A_1, A_2, A_3\}$. Thus

$$\begin{aligned} & \phi_1 \\ = & \frac{(3-1)!(1-1)!}{3!}(\nu(A_1) - \nu(\emptyset)) + \frac{(3-2)!(2-1)!}{3!}(\nu(\{A_1, A_2\}) - \nu(A_2)) \\ & + \frac{(3-2)!(2-1)!}{3!}(\nu(\{A_1, A_3\}) - \nu(A_3)) \\ & + \frac{(3-3)!(3-1)!}{3!}(\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})) \\ = & \frac{2}{6}\left(\frac{1}{4}\right) + \frac{1}{6}\left(1 - \left(-\frac{1}{3}\right)\right) + \frac{1}{6}\left(\frac{4}{3} - 0\right) + \frac{2}{6}\left(1 - \frac{3}{4}\right) \\ = & \frac{11}{18} \end{aligned}$$

Similarly, we have

$$\phi_2 = \frac{1}{36} \text{ and } \phi_3 = \frac{13}{36}$$

□

Example 5.3.5 (Used car game)**.** *Let ν be the characteristic function of the used car game (Example 5.2.8). Find the Shapley values of the players.*

Solution. Since $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$, we may use the formula

$$\begin{aligned} \phi_1 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_1, A_3\}) - 2\nu(\{A_2, A_3\})}{6} \\ &= \frac{2(700) + 500 + 700 - 2(0)}{6} \\ &= \frac{1300}{3} \end{aligned}$$

$$\begin{aligned} \phi_2 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_2, A_3\}) - 2\nu(\{A_1, A_3\})}{6} \\ &= \frac{2(700) + 500 + 0 - 2(700)}{6} \\ &= \frac{250}{3} \end{aligned}$$

$$\begin{aligned}
\phi_3 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_3\}) + \nu(\{A_2, A_3\}) - 2\nu(\{A_1, A_2\})}{6} \\
&= \frac{2(700) + 700 + 0 - 2(500)}{6} \\
&= \frac{550}{3}
\end{aligned}$$

Hence the Shapley vector is $\phi = (\frac{1300}{3}, \frac{250}{3}, \frac{550}{3})$. □

Example 5.3.6 (Mayor and council)**.** *Let ν be the characteristic function of the Mayor and council game (Example 5.2.9). Find the Shapley values of the players.*

Solution. Recall that

1. $\nu(S) = 1$ if $M \in S$ and $|S \setminus \{M\}| \geq 4$, or $|S| \geq 6$.
2. $\nu(S) = 0$ otherwise.

Thus we have

$$\phi_M = \binom{7}{4}\frac{(8-5)!(5-1)!}{8!}(1) + \binom{7}{5}\frac{(8-6)!(6-1)!}{8!}(1) = \frac{1}{4}$$

By symmetry, for each $i = 1, 2, \cdots, 7$, we have

$$\phi_i = \frac{1}{7}\left(1 - \frac{1}{4}\right) = \frac{3}{28}$$

□

Example 5.3.7 (Voting game)**.** *In a council there are 100 members. The red, blue, green, white parties has 40, 30, 25, 5 members in the council. For a resolution to pass, it is necessary to have more than 50 affirmative votes. The set of players is $A = \{R, B, G, W\}$. For any coalition $S \subset A$, define*

1. *$\nu(S) = 1$ if the total votes of S is larger than 50.*
2. *$\nu(S) = 0$ otherwise.*

Find the Shapley values of the players.

Solution. We have $\nu(R) = \nu(B) = \nu(G) = \nu(W) = 0$ and

$$\begin{cases} \nu(\{R, B\} = \nu(\{R, G\} = \nu(\{B, G\} = 1 \\ \nu(\{R, W\} = \nu(\{B, W\} = \nu(\{G, W\} = 0 \\ \nu(S) = 1 \text{ for any } S \text{ with } |S| \geq 3 \end{cases}$$

Thus

$$\begin{aligned} \phi_R &= 2\left(\frac{(4-1)!(2-1)!}{4!}\right)(1-0) + 2\left(\frac{(4-3)!(3-1)!}{4!}\right)(1-0) \\ &= \frac{1}{3} \end{aligned}$$

Similarly, we have

$$\phi_B = \phi_G = \frac{1}{3}$$
$$\phi_W = 1 - (\phi_R + \phi_B + \phi_G) = 0$$

□

The Shapley value can be defined using the axiomatic approach as follows. The Shapley vector is the unique allocation of payoffs which satisfies the 4 properties listed in the following theorem. The efficiency property requires that ϕ allocates the total worth of the grand coalition $\nu(A)$. The symmetry property asks ϕ to allocate same payoff to players with identical contributions to coalitions. The null player properties says that players who contribute nothing to every coalition should receive nothing. The linearity properties looks very natural mathematically but there is no good reason to impose such condition in the sense of fairness.

Theorem 5.3.8 (Axioms for Shapley values)**.** *The Shapley vector $\phi(\nu) = (\phi_1, \cdots, \phi_n)$ is the unique payoff allocation which satisfies the following* **axioms for Shapley values**.

1. *(Efficiency)* $\sum_{i=1}^{n} \phi_i = \nu(A)$
2. *(Symmetry) If $A_i, A_j \in A$ satisfy $\nu(S \cup \{A_i\}) = \nu(S \cup \{A_j\})$ for any coalition S not containing A_i and A_j, then $\phi_i = \phi_j$.*
3. *(Null player) If $\nu(S \cup \{A_i\}) = \nu(S)$ for any coalition S, then $\phi_i = 0$.*

4. *(Linearity) Let μ and ν be two characteristic functions and a, b be two real numbers. Then*

$$\phi(a\mu + b\nu) = a\phi(\mu) + b\phi(\nu)$$

Proof. First we prove that $\phi(\nu)$ satisfies the 4 axioms for Shapley values.

1. It has been proved in Theorem 5.3.3.

2. Suppose $\nu(S \cup \{A_i\}) = \nu(S \cup \{A_j\})$ for any coalition S not containing A_i and A_j. For any coalition $S \subset A$, denote by S' the coalition obtained by replacing A_i by A_j if $A_i \in S$ and replacing A_j by A_i if $A_j \in S$. Note that $|S'| = |S|$. We are going to prove that for any coalition S, we have

$$\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$$

First if $A_j \in S$, then $S \cup \{A_i\} = S' \cup \{A_j\}$ and thus $\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$. On the other hand, if $A_j \notin S$, then $S \setminus \{A_i\} = S' \setminus \{A_j\}$ and we also have

$$\begin{aligned} \nu(S \cup \{A_i\}) &= \nu((S \setminus \{A_i\}) \cup \{A_i\}) \\ &= \nu((S' \setminus \{A_j\}) \cup \{A_i\}) \\ &= \nu((S' \setminus \{A_j\}) \cup \{A_j\}) \\ &= \nu(S' \cup \{A_j\}) \end{aligned}$$

Thus we proved that $\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$ for any coalition S.

Therefore

$$\begin{aligned}
&\phi_i \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(\nu(S\cup\{A_i\})-\nu(S)) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S\cup\{A_i\}) - \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S'|-1)!|S'|!}{n!}\nu(S'\cup\{A_j\}) - \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S) \\
= &\sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}\nu(T\cup\{A_j\}) - \sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\
= &\sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}(\nu(T\cup\{A_j\})-\nu(T)) \\
= &\phi_j
\end{aligned}$$

3. Suppose $\nu(S\cup\{A_i\})=\nu(S)$ for any coalition S. Then

$$\begin{aligned}
\phi_i &= \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(\nu(S\cup\{A_i\})-\nu(S)) \\
&= 0
\end{aligned}$$

4. Let μ and ν be two characteristic functions and a, b be two real numbers. Then

$$\begin{aligned}
&\phi_i(a\mu+b\nu) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}((a\mu+b\nu)(S\cup\{A_i\})-(a\mu+b\nu)(S)) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(a(\mu(S\cup\{A_i\})-\mu(S))+b(\nu(S\cup\{A_i\})-\nu(S))) \\
= &a\phi_i(\mu)+b\phi_i(\nu)
\end{aligned}$$

Next we prove the uniqueness. Suppose ϕ satisfies the four axioms for Shapley values. For each non-empty coalition $S\subset A$, $S\neq\emptyset$, define a characteristic function ν_S by

$$\nu_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases}$$

Observe that if $A_i \notin S$, then $\nu_S(T \cup \{A_i\}) = \nu_S(T)$ for any $T \subset A$. Thus A_i is a null player of ν_S and we have

$$\phi_i(\nu_S) = 0 \text{ if } A_i \notin S$$

by the axiom for null player. By symmetry, we have $\phi_i(\nu_S) = \phi_j(\nu_S)$ whenever $A_i, A_j \in S$ which implies, by efficiency, that

$$\phi_i(\nu_S) = \frac{1}{|S|} \text{ if } A_i \in S$$

In conclusion we have

$$\phi_i(\nu_S) = \begin{cases} \dfrac{1}{|S|} & \text{if } A_i \in S \\ 0 & \text{if } A_i \notin S \end{cases}$$

To prove uniqueness, it suffices to prove that any characteristic function ν can be written uniquely as

$$\nu = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} c_S \nu_S$$

for some constants c_S, $S \in \mathcal{P}(A) \setminus \{\emptyset\}$. Then

$$\phi(\nu) = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} c_S \phi(\nu_S)$$

is uniquely determined. We are going to determined c_S by induction on $|S|$. Suppose $|S| = 1$, that is $S = \{A_i\}$ for some $i = 1, 2, \cdots, n$. Now for any coalition $T \subset A$, if $T = \{A_i\}$, then $T \subset S$ and $\nu_T(S) = \nu_T(A_i) = 1$. On the other hand, if $T \neq \{A_i\}$, then $\nu_T(S) = 0$. Thus for $S = \{A_i\}$, we have

$$\nu_T(S) = \nu_T(A_i) = \begin{cases} 1 & \text{if } T = \{A_i\} \\ 0 & \text{if } T \neq \{A_i\} \end{cases}$$

Hence we must have

$$\nu(A_i) = \sum_{T \in \mathcal{P}(A) \setminus \{\emptyset\}} c_T \nu_T(A_i) = c_{\{A_i\}}$$

and thus

$$c_{\{A_i\}} = \nu(A_i)$$

for $i = 1, 2, \cdots, n$. Suppose c_S is determined for each $\emptyset \neq S \subset A$ with $0 < |S| < k$. Now fix $S \subset A$ with $|S| = k$. Recall that for any coalition $T \subset A$, we have $\nu_T(S) = 1$ if $T \subset S$ and $\nu_T(S) = 0$ if T is not a subset of S. Thus we have

$$\nu(S) = \sum_{T \in \mathcal{P}(A) \setminus \{\emptyset\}} c_T \nu_T(S) = \sum_{\emptyset \neq T \subset S} c_T = c_S + \sum_{\emptyset \neq T \subsetneq S} c_T$$

and hence

$$c_S = \nu(S) - \sum_{\emptyset \neq T \subsetneq S} c_T$$

is determined because all c_T had been determined for any $\emptyset \neq T \subsetneq S$. Hence we proved that any characteristic function ν can be written uniquely as

$$\nu = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} c_S \nu_S$$

and the proof of the theorem is complete. □

In Section 5.2, we introduced the core of a cooperative game. One may ask whether the Shapley vector always lies in the core whenever the core is not empty. The answer is negative. We need an extra condition for it to be true.

Definition 5.3.9 (Convex game)**.** *We say that a characteristic function ν is* **convex** *if for any $S, T \subset A$, we have*

$$\nu(S \cup T) \geq \nu(S) + \nu(T) - \nu(S \cap T)$$

Suppose S and T are two coalitions with $T \subset S$. The contribution of $S \setminus T$ to the coalition S is $\nu(S) - \nu(T)$. In a convex game, this contribution of $S \setminus T$ cannot be larger if the coalition T gets smaller. More precisely, we have

Theorem 5.3.10. *Suppose* ν *is a convex game. For any coalitions* R, S, T *with* $R \subset T \subset S$*, we have*

$$\nu(S) - \nu(T) \geq \nu(S \setminus R) - \nu(T \setminus R)$$

Proof. Consider $S = (S \setminus R) \cup T$. By convexity of ν, we have

$$\begin{aligned} \nu(S) &= \nu((S \setminus R) \cup T) \\ &\geq \nu(S \setminus R) + \nu(T) - \nu((S \setminus R) \cap T) \\ &= \nu(S \setminus R) + \nu(T) - \nu(T \setminus R) \end{aligned}$$

□

Now we can prove

Theorem 5.3.11. *The Shapley vector of a convex game always lies in the core. In particular, the core of a convex game is not empty.*

Proof. Let ν be a convex game with player set $A = \{1, 2, \cdots, n\}$ and ϕ be the Shapley vector of ν. For any permutation $\sigma \in S_n$, define $\phi^\sigma = (\phi_1^\sigma, \phi_2^\sigma, \cdots, \phi_n^\sigma) \in \mathbf{R}^n$ with

$$\phi_k^\sigma = \nu(S_k^\sigma) - \nu(S_k^\sigma \setminus \{k\})$$

where $S_k^\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(i)\}$ and i is the integer determined by $\sigma(i) = k$. We have seen (Theorem 5.3.2) that

$$\phi = \frac{1}{n!} \sum_{\sigma \in S_n} \phi^\sigma$$

Since the core $C(\nu)$ is convex, it suffices to prove that $\phi^\sigma \in C(\nu)$ for any $\sigma \in S_n$. Without loss of generality, we may assume that σ is the identity, that is, $\sigma(k) = k$ for any $k =, 1, 2, \cdots, n$. In this case, for any coalition $S = \{s_1 < s_2 < \cdots < s_m\} \subset A$ and $s_i \in S$, $i = 1, 2, \cdots, m$, we have $S_{s_i}^\sigma = \{1, 2, \cdots, s_i\}$ and

$$\begin{aligned} \phi_{s_i}^\sigma &= \nu(S_{s_i}^\sigma) - \nu(S_{s_i}^\sigma \setminus \{s_i\}) \\ &= \nu(\{1, 2, \cdots, s_i\}) - \nu(\{1, 2, \cdots, s_i - 1\}) \\ &\geq \nu(\{s_1, s_2, \cdots, s_i\}) - \nu(\{s_1, s_2, \cdots, s_{i-1}\}) \end{aligned}$$

where in the last line, we removed those elements not in S in both sets and the inequality follows from Theorem 5.3.10. Thus

$$\begin{aligned} \sum_{s_i \in S} \phi^{\sigma}_{s_i} &= \sum_{s_i \in S} (\nu(S^{\sigma}_k) - \nu(S^{\sigma}_k \setminus \{k\})) \\ &\geq \sum_{s_i \in S} (\nu(\{s_1, s_2, \cdots, s_i\}) - \nu(\{s_1, s_2, \cdots, s_{i-1}\})) \\ &= \nu(\{s_1, s_2, \cdots, s_m\}) - \nu(\emptyset) \\ &= \nu(S) \end{aligned}$$

Hence we have $\phi^{\sigma} \in C(\nu)$ by Theorem 5.2.4. Therefore $\phi = \frac{1}{n!} \sum_{\sigma \in S_n} \phi^{\sigma} \in C(\nu)$ since $C(\nu)$ is a convex set. □

As a matter of fact, Shapley proved that if ν is convex, then $C(\nu)$ is a convex polyhedron of dimension $n-1$ with $2^n - 2$ faces and $n!$ vertices located exactly at ϕ^{σ}'s, $\sigma \in S_n$. Therefore the Shapley vector ϕ is precisely the center of mass of the vertices of the core when ν is convex.

Exercise 5

1. Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.

Strategy	Payoff vector
$(0,0,0)$	$(-2,3,5)$
$(0,0,1)$	$(1,-2,7)$
$(0,1,0)$	$(1,5,0)$
$(0,1,1)$	$(10,-3,-1)$
$(1,0,0)$	$(-1,0,7)$
$(1,0,1)$	$(-4,4,6)$
$(1,1,0)$	$(12,-4,-2)$
$(1,1,1)$	$(-1,5,2)$

   (a) Find the characteristic function of the game.

   (b) Show that the core of the game is empty.

2. Consider a three-person game with characteristic function

$$\begin{aligned} \nu(\{1\}) &= 27 \\ \nu(\{2\}) &= 8 \\ \nu(\{3\}) &= 18 \\ \nu(\{1,2\}) &= 36 \\ \nu(\{1,3\}) &= 50 \\ \nu(\{2,3\}) &= 27 \\ \nu(\{1,2,3\}) &= 60 \end{aligned}$$

   Find the core of the game and draw the region representing the core on the $x_1 - x_2$ plane.

3. Let ν be the characteristic function defined by $\nu(\{1\}) = 3, \nu(\{2\}) = 4, \nu(\{3\}) = 6, \nu(\{1,2\}) = 9, \nu(\{1,3\}) = 12, \nu(\{2,3\}) = 15, \nu(\{1,2,3\}) = 20$.

   (a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{1,2\}), \mu(\{1,3\}), \mu(\{2,3\})$.

   (b) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

   (c) Find the Shapley values of the players.

4. Three towns A, B, C are considering whether to built a joint water distribution system. The costs of the construction works are listed in the following table

Coalition	Cost(in million dollars)
$\{A\}$	11
$\{B\}$	7
$\{C\}$	8
$\{A, B\}$	15
$\{A, C\}$	14
$\{B, C\}$	13
$\{A, B, C\}$	20

   For any coalition $S \subset \{A, B, C\}$, define $\nu(S)$ to be the amount saved if they build the system together. Find the Shapley values of A, B, C and the amount that each of them should pay if they cooperate.

5. Players 1, 2, 3 and 4 have 45, 25, 15, and 15 votes respectively. In order to pass a certain resolution, 51 votes are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain resolution. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

6. Players 1, 2, 3 and 4 have 40, 30, 20, and 10 shares if stocks respectively. In order to pass a certain decision, 50 shares are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain decision. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

7. Consider the following market game. Each of the 5 players starts with one glove. Two of them have a right-handed glove and three of them have a left-handed glove. At the end of the game, an assembled pair is worth \$1 to whoever holds it. Find the Shapley value of the players.

8. Let $\mathcal{A} = \{1, 2, 3\}$ be the set of players and ν be a game in characteristic form with

$$\begin{aligned}
\nu(\{1\}) &= -a \\
\nu(\{2\}) &= -b \\
\nu(\{3\}) &= -c \\
\nu(\{2,3\}) &= a \\
\nu(\{1,3\}) &= b \\
\nu(\{1,2\}) &= c \\
\nu(\{1,2,3\}) &= 1
\end{aligned}$$

where $0 \leq a, b, c \leq 1$.

   (a) Let μ be the $(0, 1)$ reduced form of ν. Find $\mu(\{1, 2\})$, $\mu(\{1, 3\})$, $\mu(\{2, 3\})$ in terms of a, b, c.

   (b) Suppose $a + b + c = 2$. Find an imputation $\mathbf{x}$ of ν which lies in the core $C(\nu)$ in terms of a, b, c and prove that $C(\nu) = \{\mathbf{x}\}$.

9. Consider an airport game which is a cost allocation problem. Let $N = \{1, 2, \cdots, n\}$ be the set of players. For each $i = 1, 2, \cdots, n$, player i requires an airfield that costs c_i to build. To accommodate all the players, the field will be built at a cost of $\max_{1 \leq i \leq n} c_i$. Suppose all

the costs are distinct and $c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be

$$\nu(S) = -\max_{i \in S} c_i$$

For each $k = 1, 2, \cdots, n$, let $R_k = \{k, k+1, \cdots, n\}$ and define

$$\nu_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset \\ 0 & \text{if } S \cap R_k = \emptyset \end{cases}$$

(a) Show that $\nu = \sum_{k=1}^{n} \nu_k$

(b) Show that for each $k = 1, 2, \cdots, n$, if $i \notin R_k$, then player i is a null player of ν_k.

(c) Show that for each $k = 1, 2, \cdots, n$, if $i, j \in R_k$, then player i and player j are symmetric players of ν_k.

(d) Find the Shapley value $\phi_k(\nu)$ of player k, $k = 1, 2, \cdots, n$, of the airport game ν.

10. Let $\mathcal{A} = \{1, 2, \cdots, N\}$. Prove that for any $i \in \mathcal{A}$

$$\sum_{\{i\} \subset S \subset \mathcal{A}} (N - |S|)!(|S| - 1)! = N!$$

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